Matched Interface and Boundary Method for Solving the Heat Equation with Interfaces

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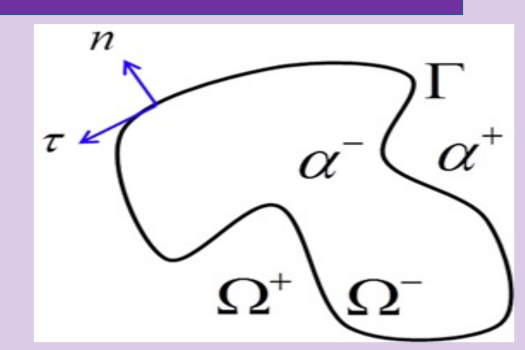
Abstract

When modeling systems made up of two materials with different thermodynamic properties, a physical interface can be introduced to account for the border where the materials meet. This interface separates our model's standard grid into two regions, each with its unique physical properties. At these interfaces, boundary conditions can be imposed to represent the difference in heat and in heat flux between the different materials so that their interaction may be modeled accurately. Because standard finite difference methods are inadequate to deal with interfaces, a Matched Interface and Boundary (MIB) technique is investigated in this work to solve the heat equation with interfaces. MIB techniques are powerful tools used to solve partial differential equations due to their efficiency and stability [4]. Without loss of generality, this work will solve 1-dimensional interface problems to demonstrate the accuracy and computational efficiency of this method, which will create a linear system of equations to be solved at each step in time throughout the duration of the model.

Governing Model of the Interface Problem

The heat equation for two distinct domains by an interface Γ is given by the following

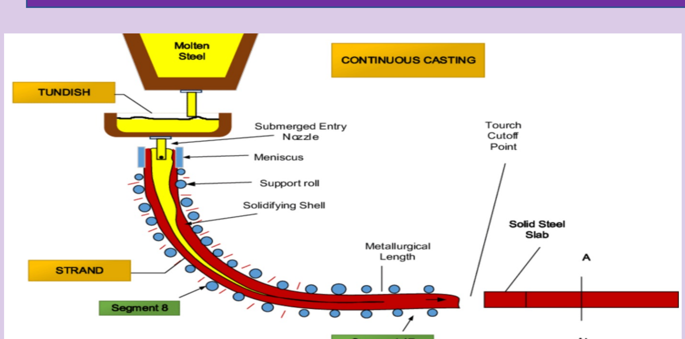
$$u_t = \nabla(\alpha \nabla u) + f \qquad \alpha = \begin{cases} \alpha^+ & in\ \Omega^+ \\ \alpha^- & in\ \Omega^- \end{cases} \qquad f = \begin{cases} f^+ & in\ \Omega^+, \\ f^- & in\ \Omega^- \end{cases} \tag{1}$$
$$\Omega = \Omega^- \cup \Omega^+$$
$$\Gamma = \Omega^- \cap \Omega^+$$



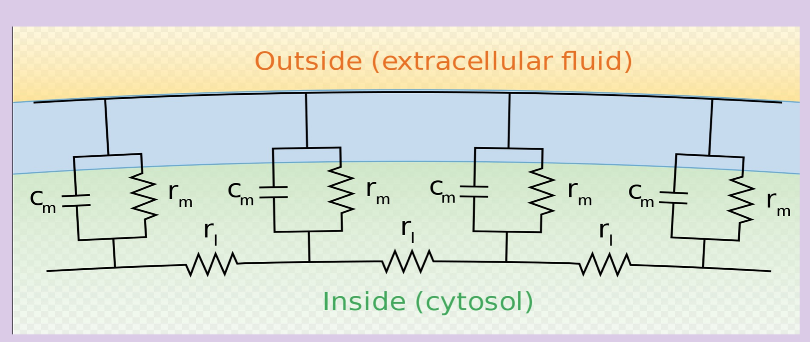
Where u(x,t) is the temperature at a given time, α is the diffusion coefficient, Ω is a square domain containing Ω^+ and Ω^- and f is the source term. The jump conditions are as follows

$$[u] = u^+ - u^- = \phi(s,t), \qquad [\alpha u_n] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi(s,t) \tag{2}$$

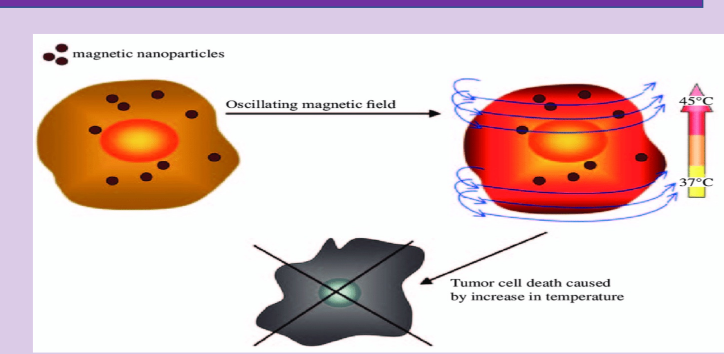
Applications



The Transient Heat Conduction Equation [1]



The Cable Equation [3]



Penn's Bioheat Equation [2]

Demonstration of Methods in 2 Dimensions

- **Temporal Discretization: Backward Euler Implicit Method**

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\Delta t} = \alpha\left(\frac{u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j+1}^{k+1}}{h^2} + \frac{u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1}}{h^2}\right) + f_{i,j}^{k+1}, \tag{3}$$

Where Δt is the time step and Δx and Δy are spatial steps.

$$\left(\frac{1}{\alpha} + 4\frac{\Delta t}{h^2}\right) u_{i,j}^{k+1} - \frac{\Delta t}{h^2}\left(u_{i,j-1}^{k+1} + u_{i,j+1}^{k+1} + u_{i-1,j}^{k+1} + u_{i+1,j}^{k+1}\right) = \frac{1}{\alpha} u_{i,j}^{k} + \frac{\Delta t}{\alpha} f_{i,j}^{k+1}, \tag{4}$$

$\Delta x = \Delta y = h$ Since we are using a square grid.

- **Spatial Discretization: Matched Interface and Boundary Method**

-The standard central difference formula for grids facing away from the interface

$$\delta_{yy} u_{i,j}^{k} := \frac{1}{h^2}\left(u_{i,j-1}^{k} - 2u_{i,j}^{k} + u_{i,j+1}^{k}\right) \tag{5}$$
$$\delta_{xx} u_{i,j}^{k} := \frac{1}{h^2}\left(u_{i-1,j}^{k} - 2u_{i,j}^{k} + u_{i+1,j}^{k}\right) \tag{6}$$

References

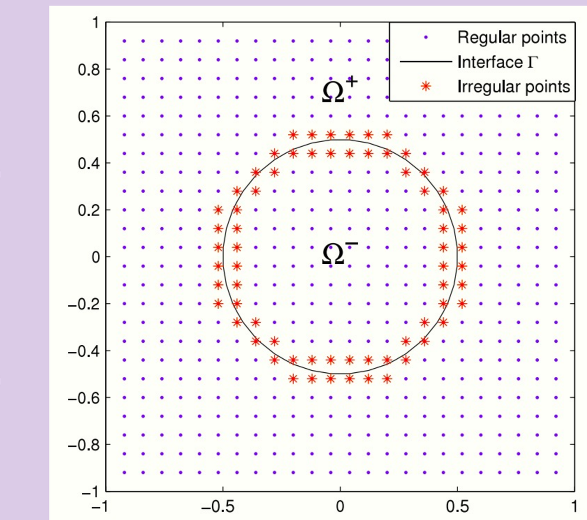
1. Alizadeh, Mehdi & Edris, Hossein & Shafyei, Ali. (2006). Mathematical Modeling of Heat Transfer for Steel Continuous Casting Process. International Journal of ISSI. 3.
2. Giordano MA, Gutierrez G, Rinaldi C. Fundamental solutions to the bioheat equation and their application to magnetic fluid hyperthermia. Int J Hyperthermia. 2010;26(5):475-84. doi: 10.3109/02656731003749643. PMID: 20578812.
3. Holmes W.R. (2014) Cable Equation. In: Jaeger D., Jung R. (eds) Encyclopedia of Computational Neuroscience. Springer, New York, NY. https://doi.org/10.1007/978-1-4614-7320-6_478-1
4. Zhao S. (2015) A Matched Alternating Direction Implicit (ADI) Method for Solving the Heat Equation with Interfaces. Journal of Scientific Computing (2015) 63: 118. doi:10.1007/s10915-014-9887-0

Demonstration of Methods in 2 Dimension Continued

- **Spatial Discretization: Matched Interface and Boundary Method (continued)**

-Decomposition of the jump conditions to x and y directions

$$\frac{\partial}{\partial n} = cos(\theta)\frac{\partial}{\partial x} + sin(\theta)\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \tau} = -sin(\theta)\frac{\partial}{\partial x} + cos(\theta)\frac{\partial}{\partial y} \tag{7}$$

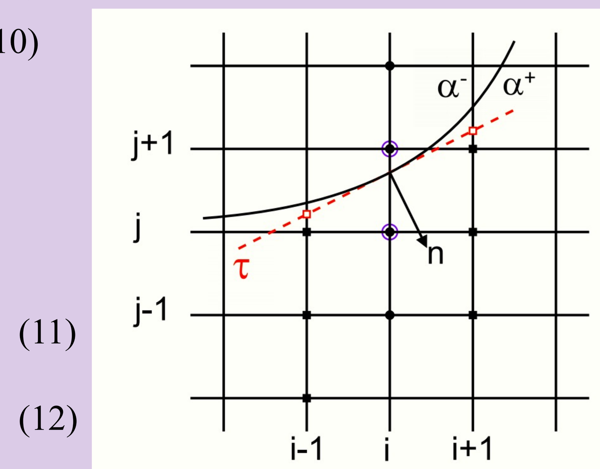


- Here, however, we can see an obvious coupling of the x and y directions incorporated into the flux jump conditions

$$cos(\theta)\,[\alpha u_x] + \sin(\theta)\,[\alpha u_y] = \psi \tag{8}$$

- Giving us the derived jump conditions

$$[\alpha u_x] = \psi cos\theta - sin\theta(\alpha^+ - \alpha^-)\mathbf{u}_\tau^+ - sin\theta[\alpha^- \phi_\tau] := \bar{\psi} \tag{9}$$
$$[\alpha u_y] = \psi sin\theta + cos\theta(\alpha^+ - \alpha^-)\mathbf{u}_\tau^+ - cos\theta[\alpha^- \phi_\tau] := \hat{\psi} \tag{10}$$

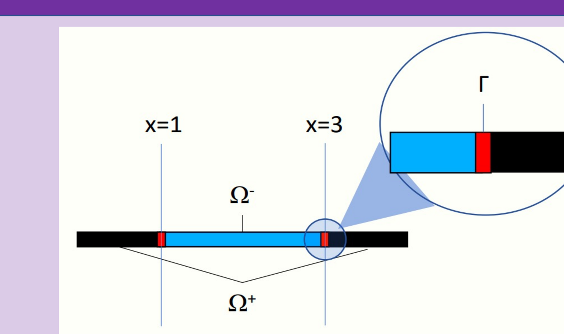


-To modify the central difference formula in the MIB scheme at irregular nodes immediately juxtaposed with the interface Γ in the MIB scheme.

$$\delta_{yy} u_{i,j}^{k+1} = \frac{1}{h^2}\left(u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + \tilde{u}_{i,j+1}^{k+1}\right), \; \delta_{yy} u_{i,j+1}^{k+1} = \frac{1}{h^2}\left(\tilde{u}_{i,j}^{k+1} - 2u_{i,j+1}^{k+1} + u_{i,j+2}^{k+1}\right) \tag{11}$$
$$\delta_{xx} u_{i,j}^{k+1} = \frac{1}{h^2}\left(u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + \tilde{u}_{i+1,j}^{k+1}\right), \; \delta_{xx} u_{i+1,j}^{k+1} = \frac{1}{h^2}\left(\tilde{u}_{i,j}^{k+1} - 2u_{i+1,j}^{k+1} - u_{i+2,j}^{k+1}\right) \tag{12}$$

where $\tilde{u}_{i,j}^{k+1}$ is considered for both the x and y directions independently and $\tilde{u}_{i+1,j}^{k+1}$, and $\tilde{u}_{i,j+1}^{k+1}$ are additional "fictitious values" that represent the approximation of nodes on the opposite side of the interface.

Numerical Experiment in 1 Dimension



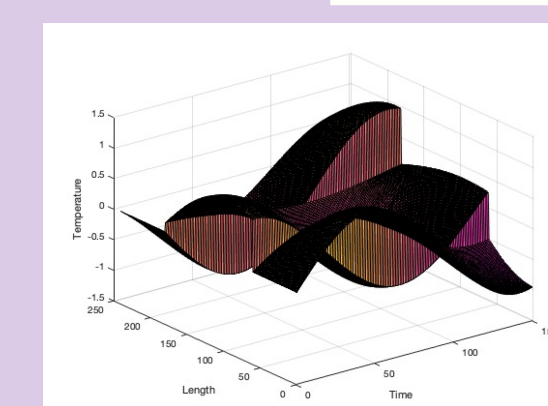
- **Example 1**

The analytical solution is given as

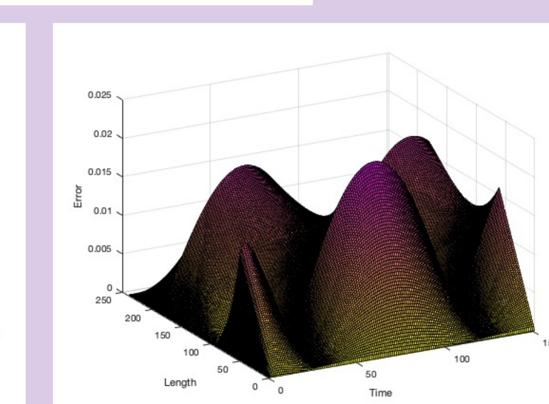
$$u(x,t) = \begin{cases} cos(x)\,sin(t) & \Omega^+ \\ sin(x)\,cos(t) & \Omega^- \end{cases}$$

With the jump conditions

$$\phi = cos(x)\,sin(t) - sin(x)\,cos(t)$$
$$\psi = -10\,sin(x)\,sin(t) - cos(x)\,cos(t)$$



Time interval = [0,5]
Number of time steps = 150
Space interval = [0,4]



Absolute Error

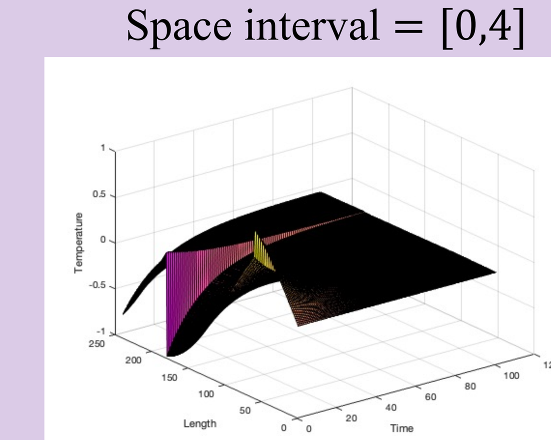
- **Example 2**

The analytical solution is given as

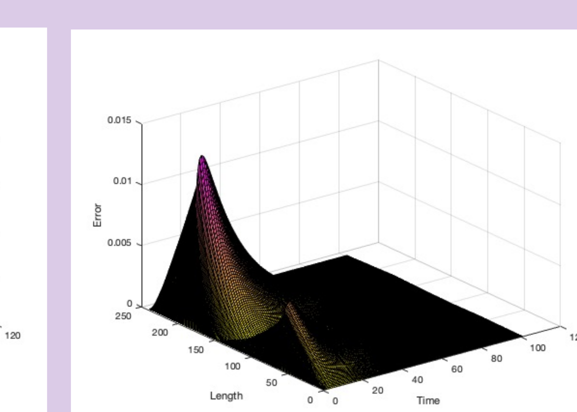
$$u(x,t) = \begin{cases} e^{-t}\,sin(x) & \Omega^+ \\ e^{-t}\,cos(x) & \Omega^- \end{cases}$$

With the jump conditions

$$\phi = e^{-t}\,sin(x) - e^{-t}cos(x)$$
$$\psi = 10e^{-t}\,cos(x) + e^{-t}\,sin(x)$$



Time interval = [0,5]
Number of time steps = 100
Space interval = [0,4]



Absolute Error

Conclusion

Since the method is shown to be accurate for the example cases given, we can now use this method to get solutions to problems where the analytical solution is unknown. Further improvements can be made going forward from the current investigation. To increase the number of spatial dimensions used, an Alternating Direction Implicit method can be used in conjunction with the MIB method. Making these improvements allows for more complex models to be used in the future, which can be used in a number of applications.