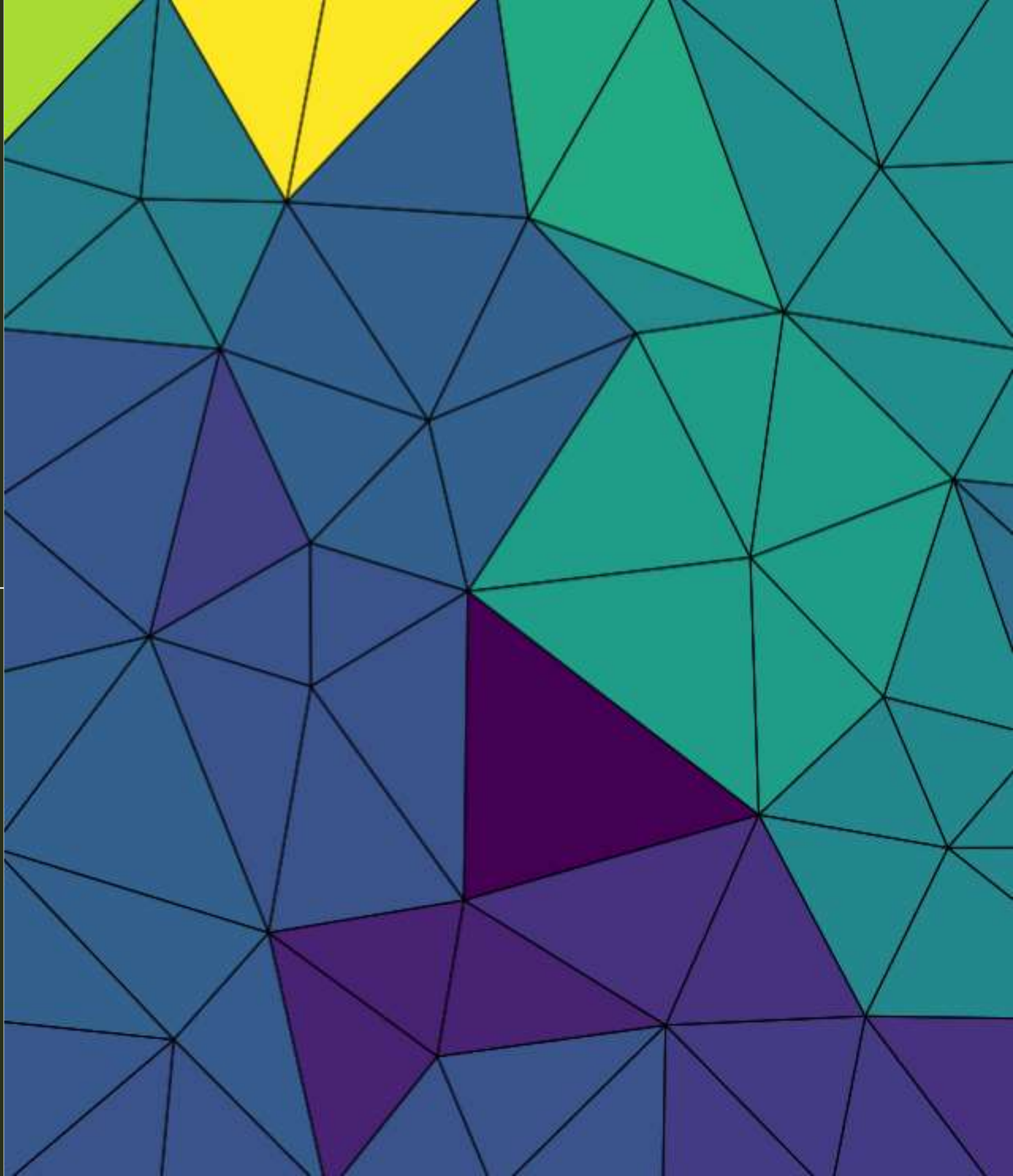
Solving Parabolic Interface Problems with a Finite Element Method

Henry Brown



WCU Research & Creative Activity Day

4-29-2021

Parabolic Interface PDEs

- Defined on a which is split by an interface.
- Solutions may be discontinuous over the interface.

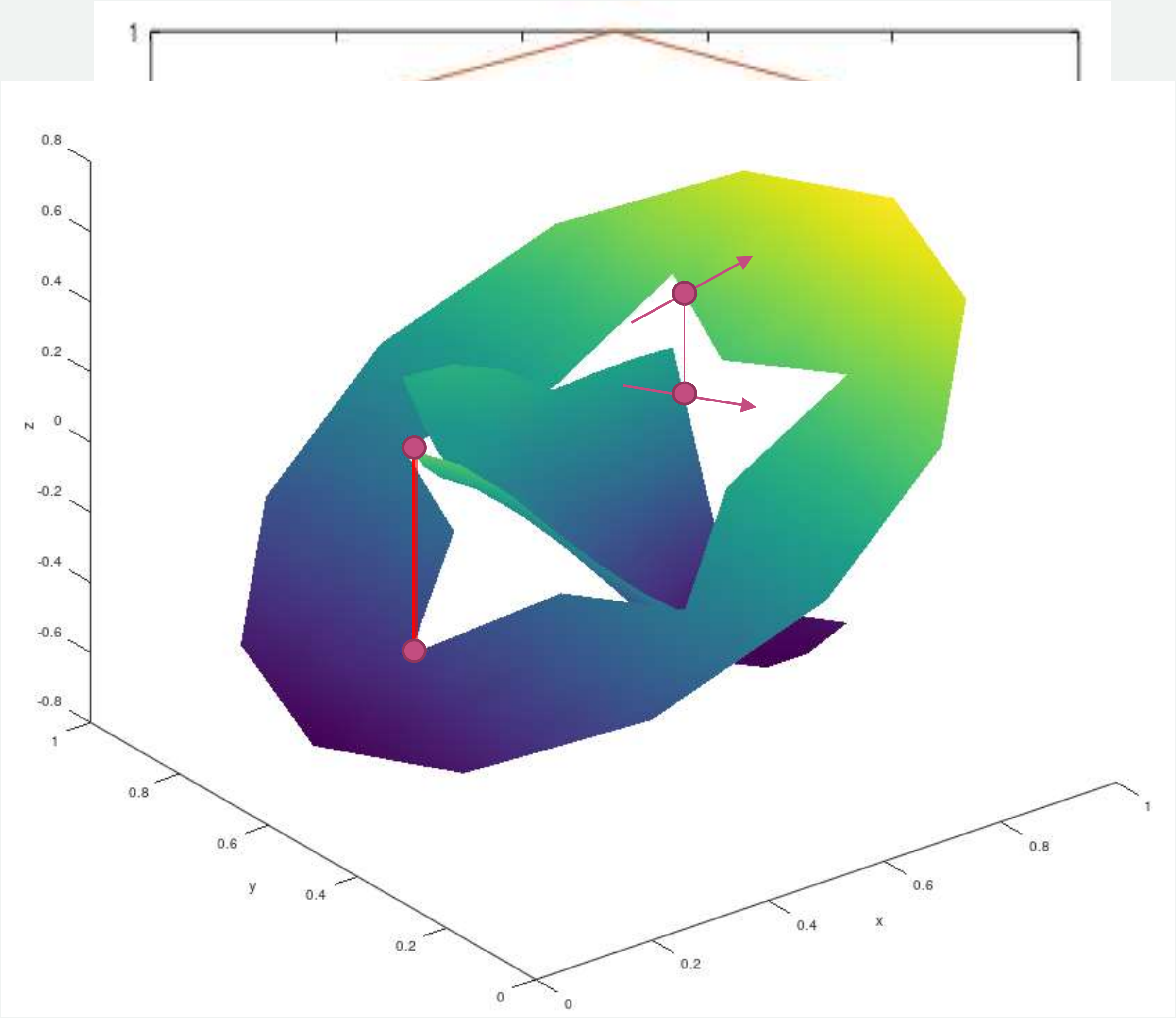
$$\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f,\ x \in \Omega$$

$$u = g_D, x \in \partial\Omega_D$$

$$\alpha \frac{\partial u}{\partial n} = g_N,\ x \in \partial\Omega_N$$

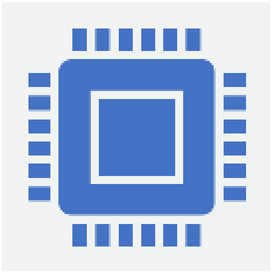
$$[u] = u^+ - u^- = \phi,\ x \in \Gamma$$

$$[\alpha \frac{\partial u}{\partial n}] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi,\ x \in \Gamma$$



Why Computational Methods?

Difficultly/impossibility of finding closed form analytical solutions

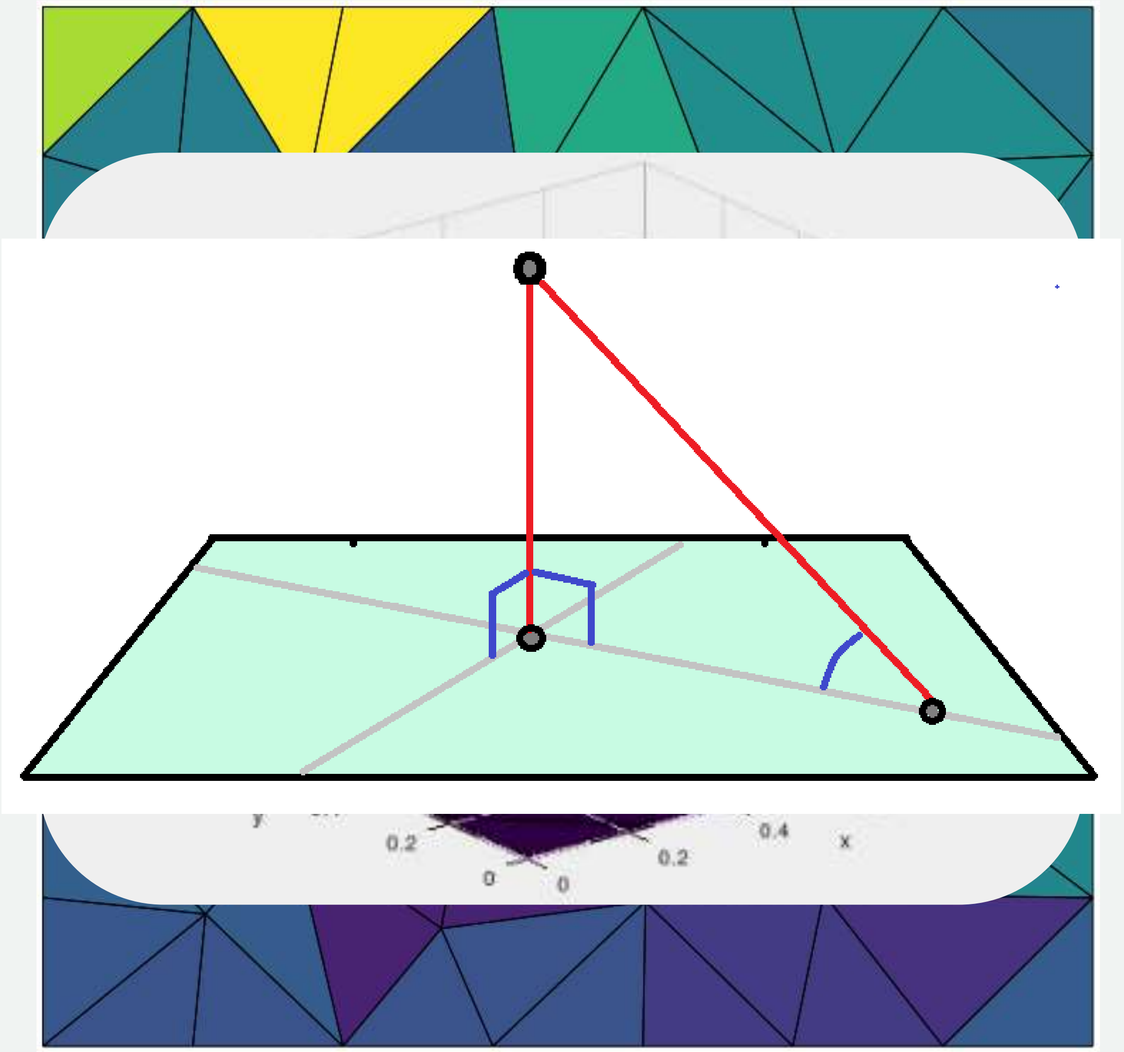


Handle a varying array problems without added complexity

These methods have proven error bounds and convergence

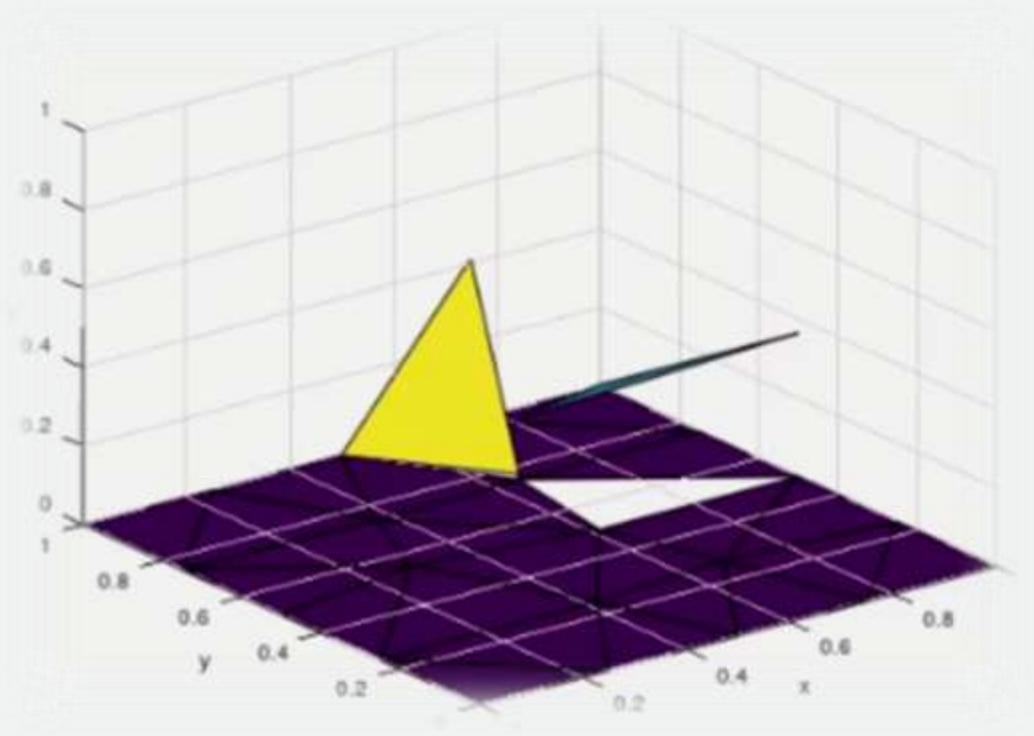
Finite Element Methods (FEMs)

- Can use non-uniform meshes
- Piecewise polynomials over the elements using basis functions
- Projection theorems ensure that FEM finds the best approximation in the function space



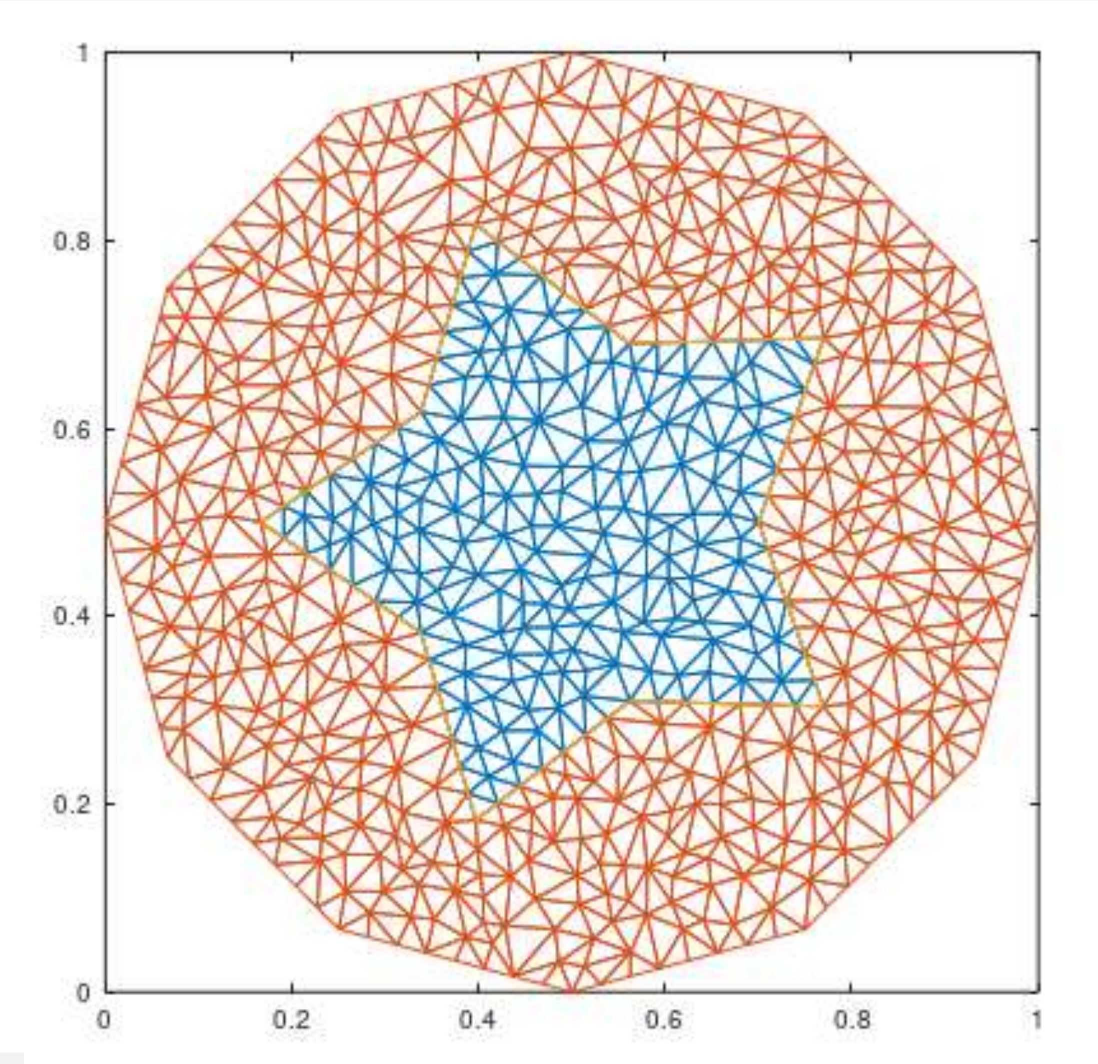
Discontinuous Galerkin (DG) FEMs

- Allows for discontinuity over element boundaries
- Broken piecewise polynomials over the elements
- A natural means to apply the jump conditions in the parabolic interface problems



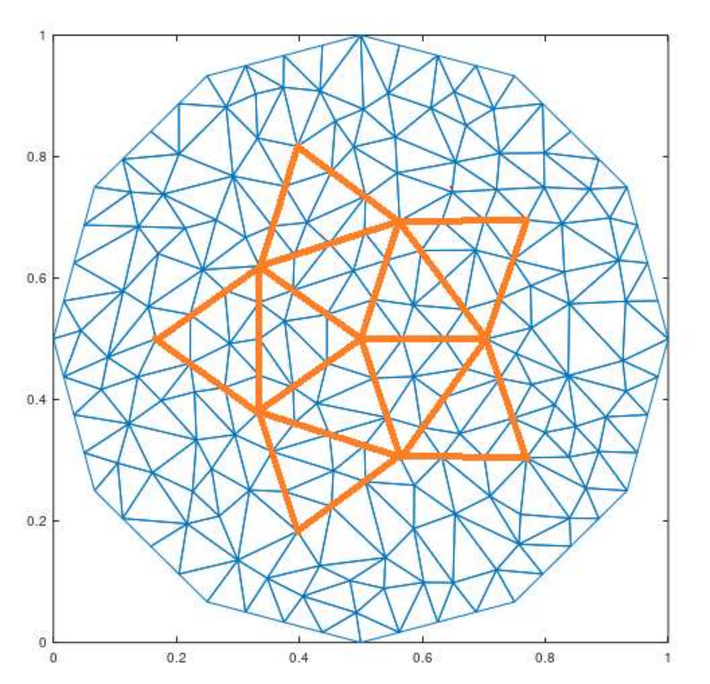
Creating a Conforming Triangulation

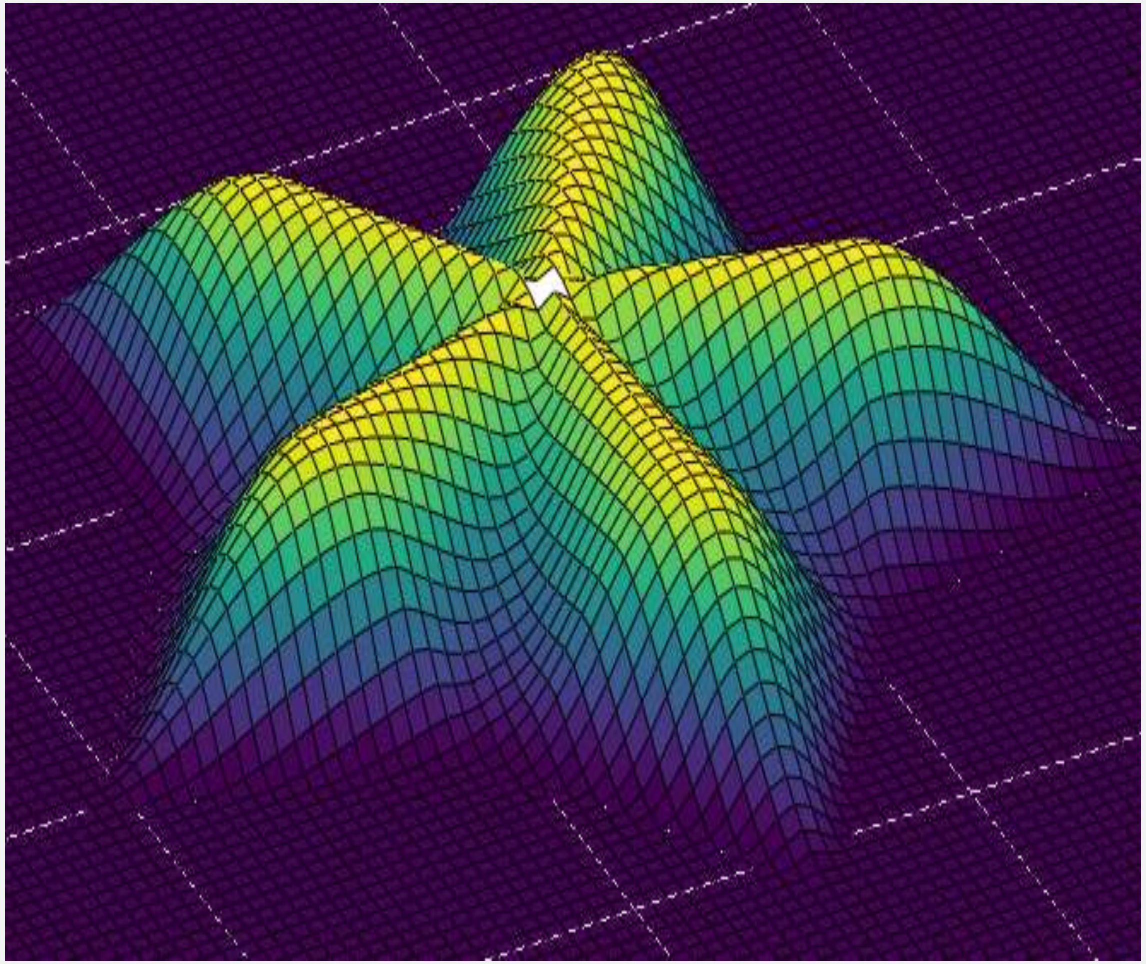
- Triangulation which conforms to the interface
- Triangulation can be refined, which gives room more accurate solution



First Test Problem (Starfish)

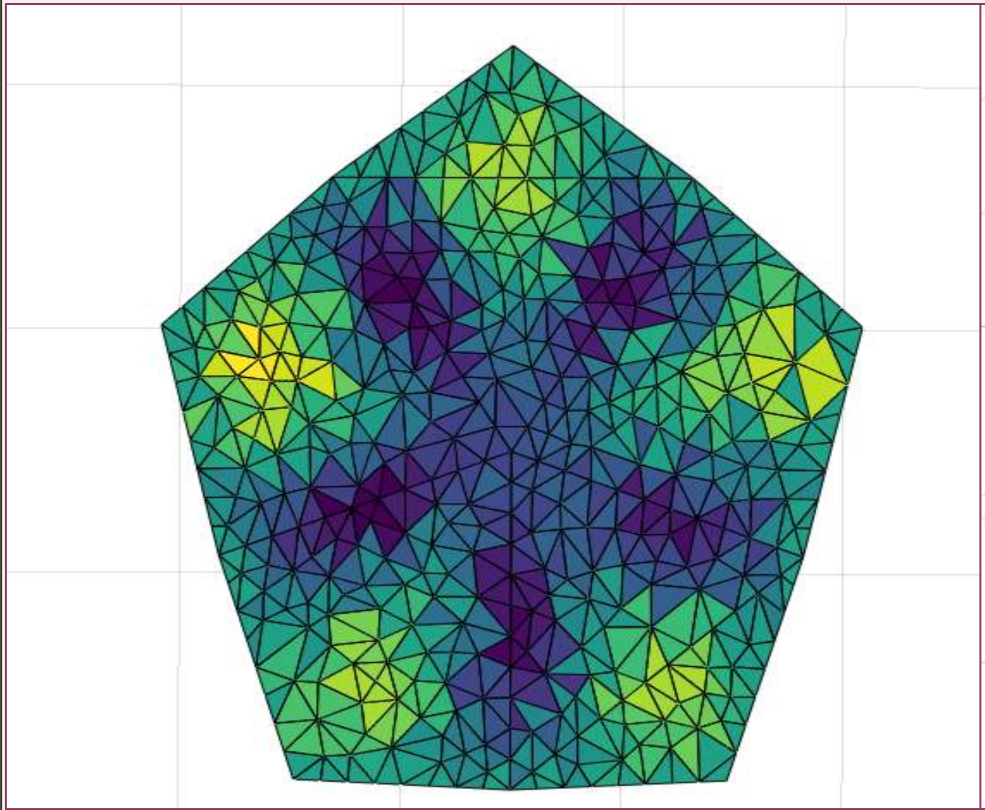
- Zero if not in the star
- Piecewise source term
- Initial Condition (Right)

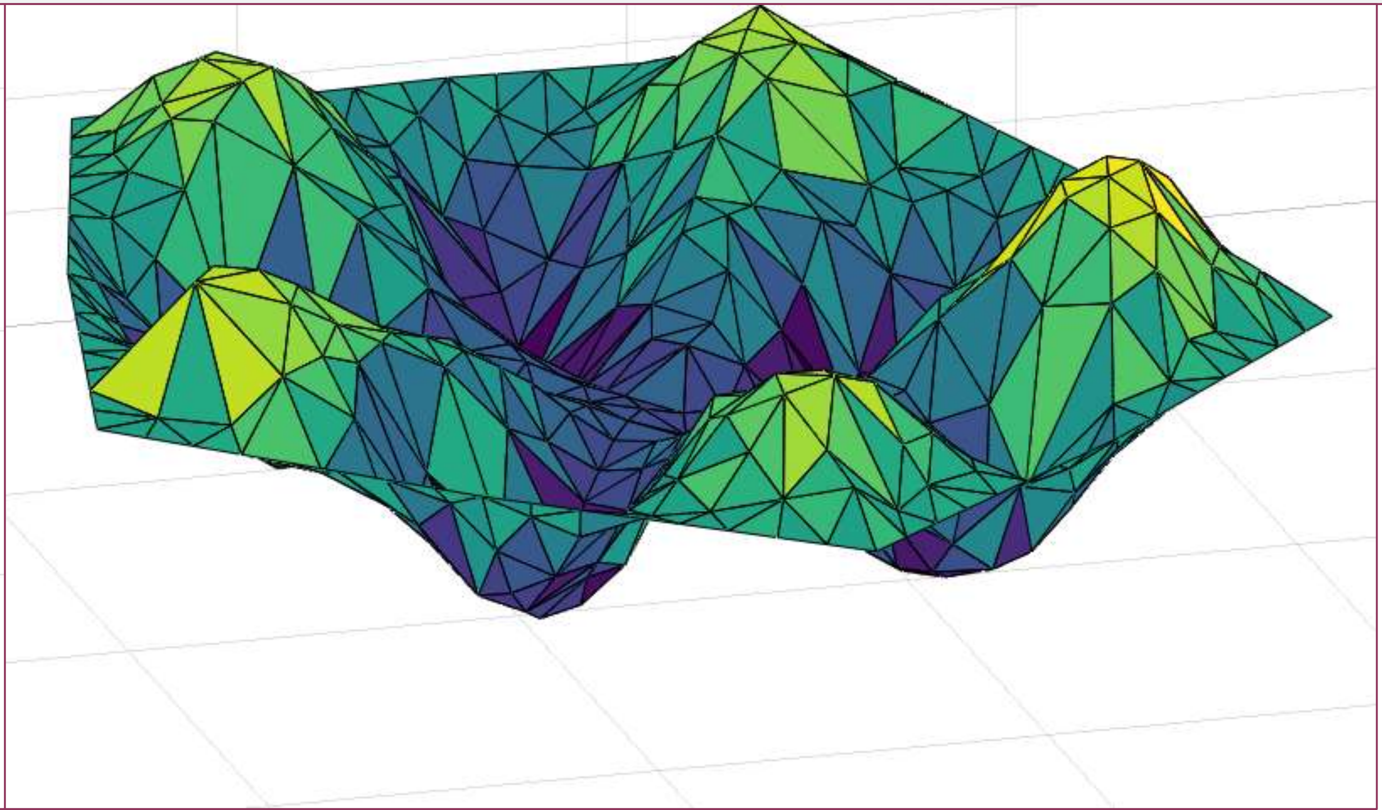




Issues with the Test Problem

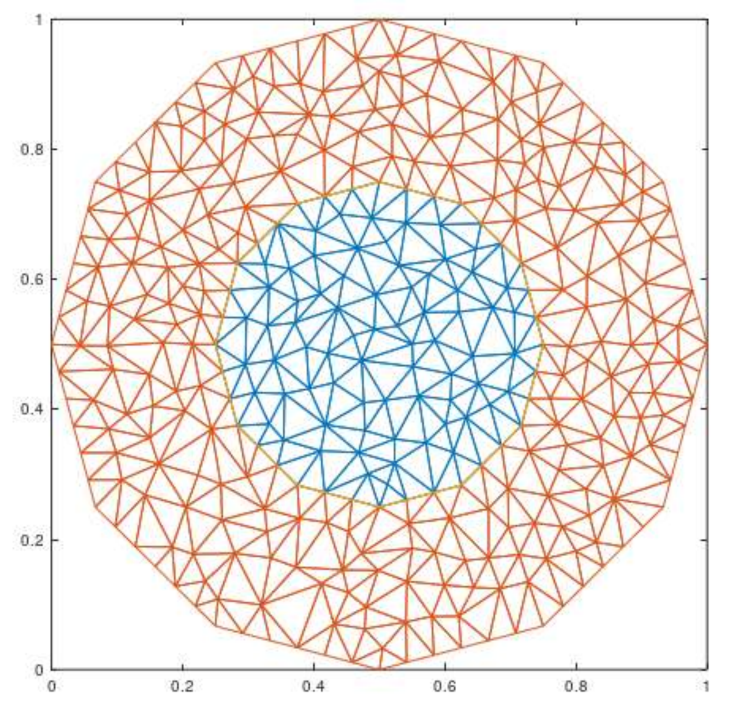
- Sinking behavior
- Trouble with the center

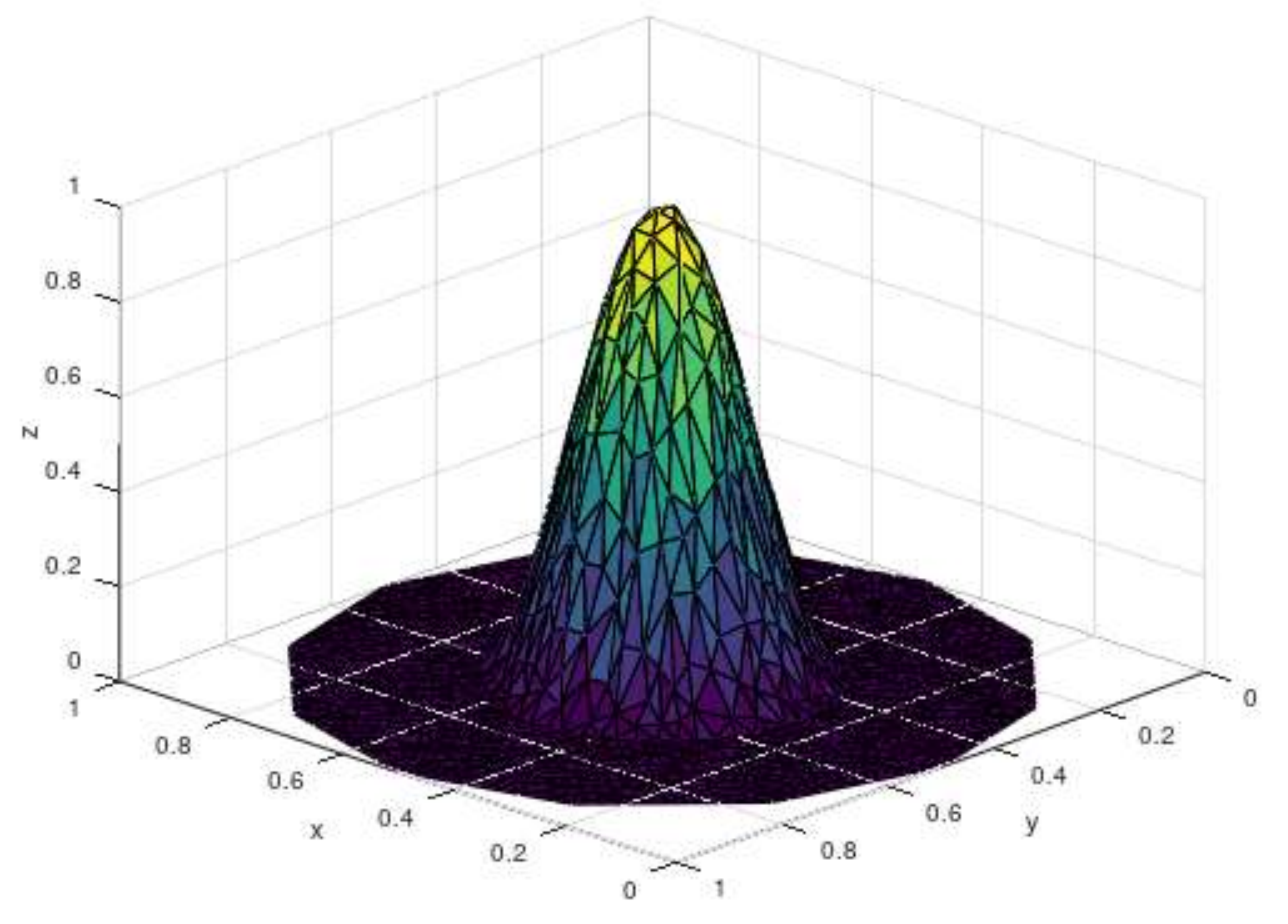




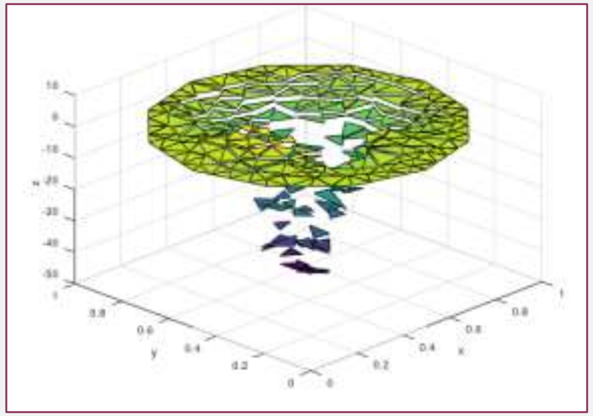
Changing the Test Problem

- Less complex geometry
- Initial Condition (Right)

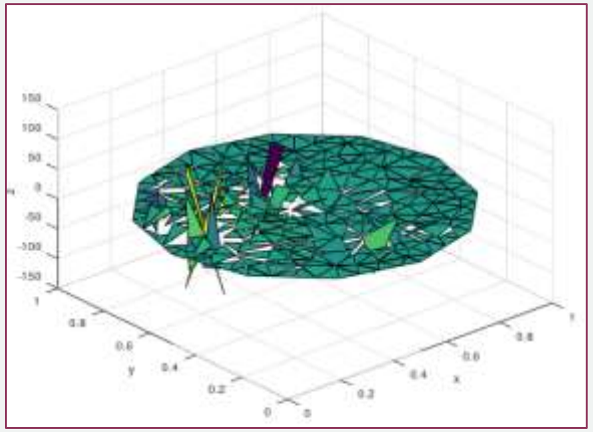




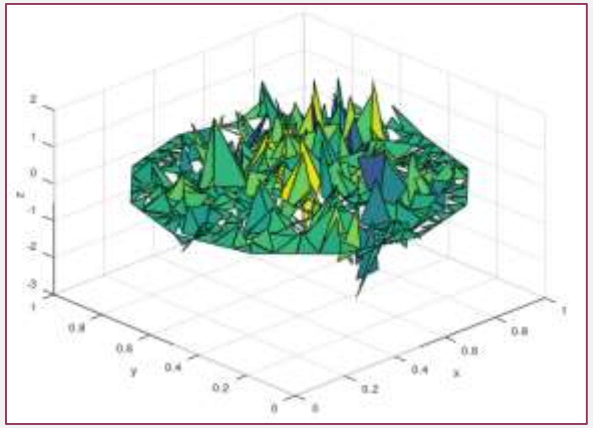
Stability and Over-Penalizing the Interface



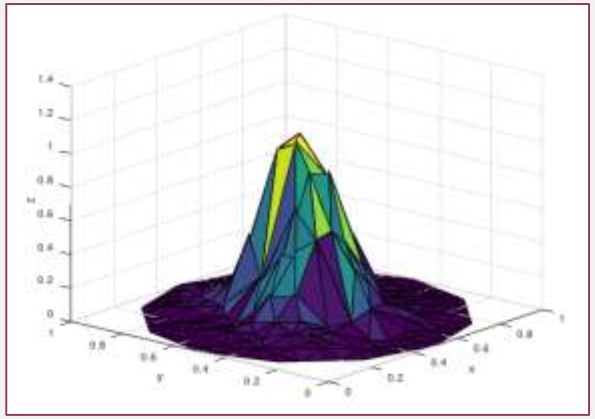
Penalty: 800



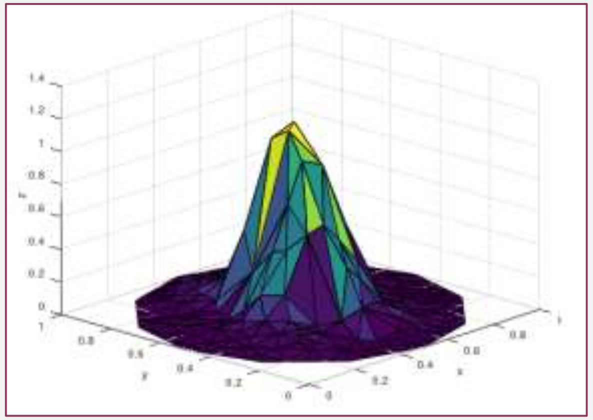
Penalty: 8000



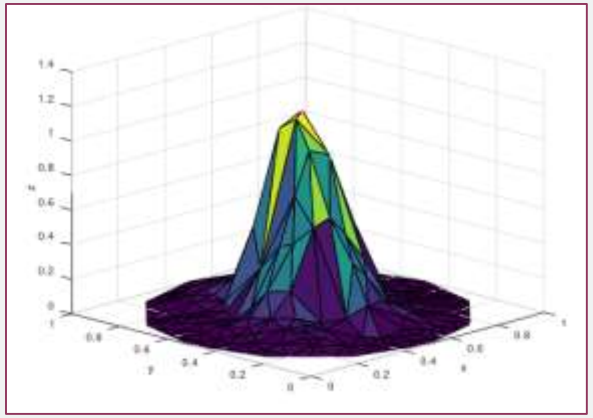
Penalty: 80000



Penalty: 800000

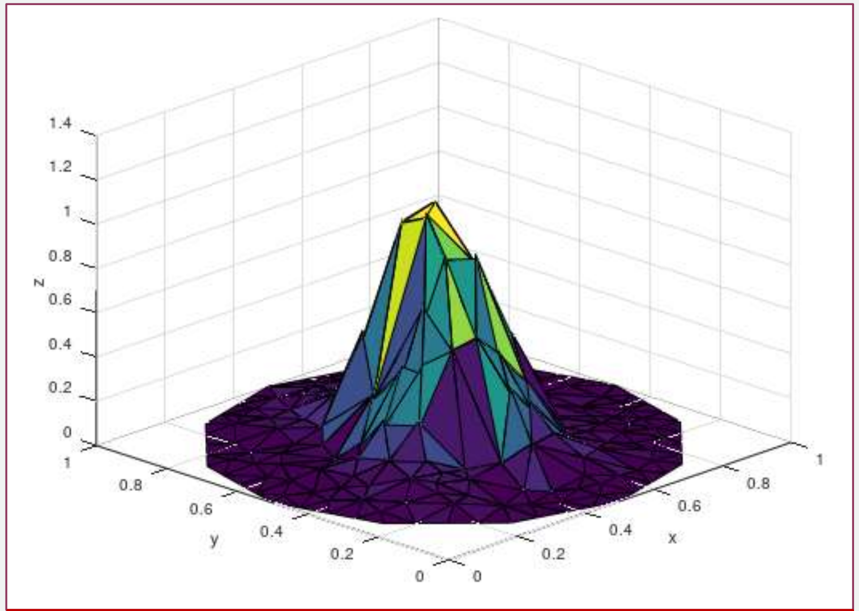


Penalty: 8000000

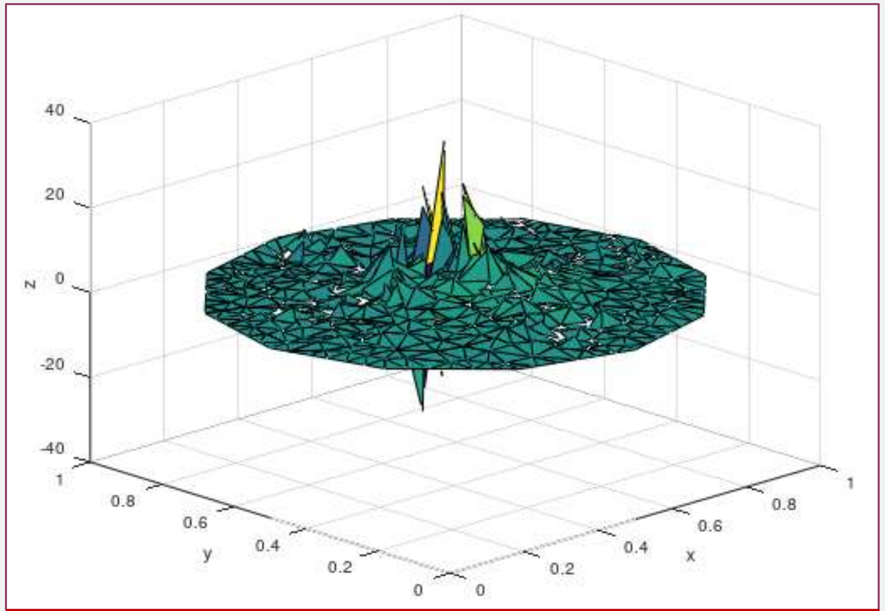


Penalty: $8x10^{13}$

Penalty Dependence on Mesh



Density: 300

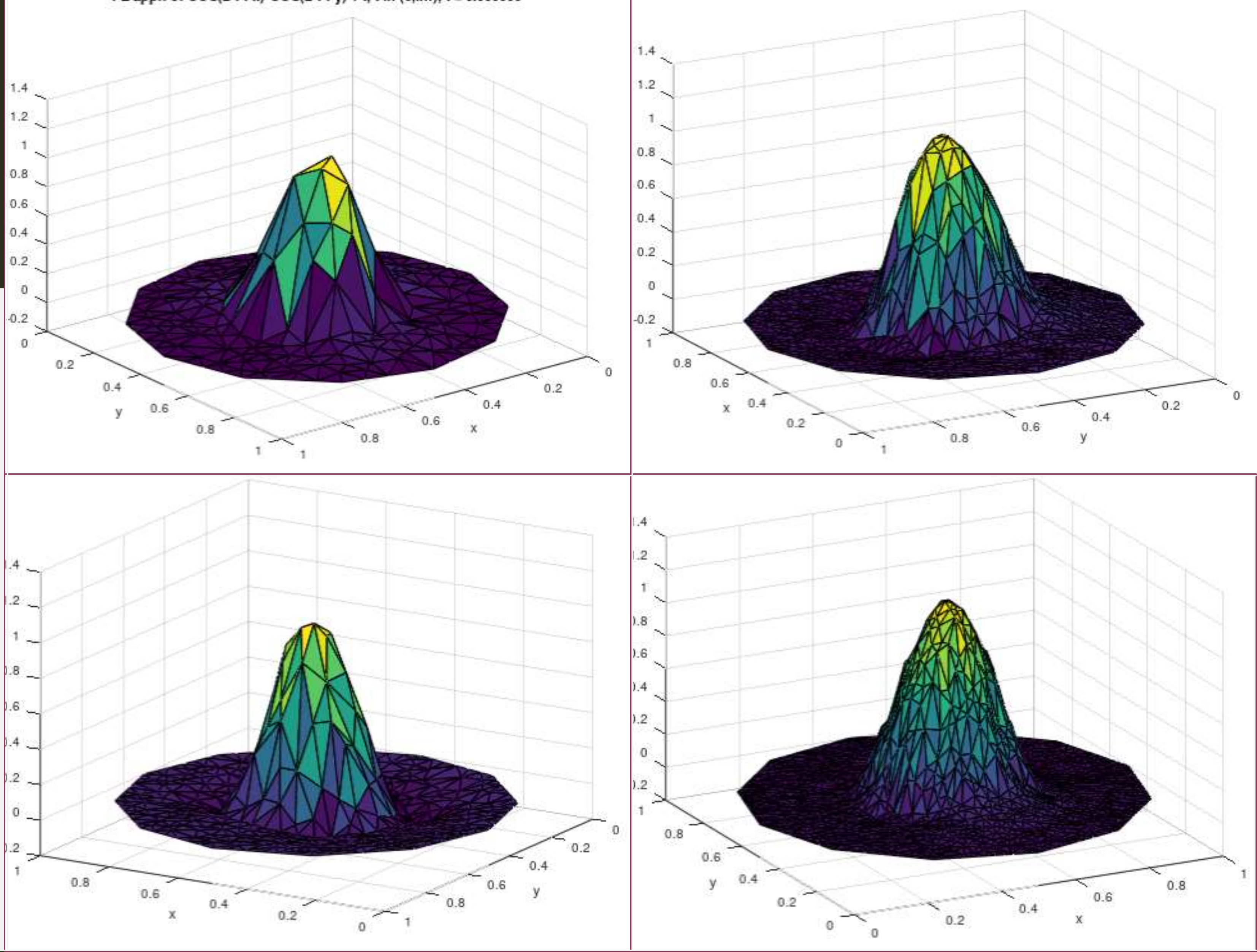


Density: 600

- Penalty: 800000
- Optimally, no dependence on mesh

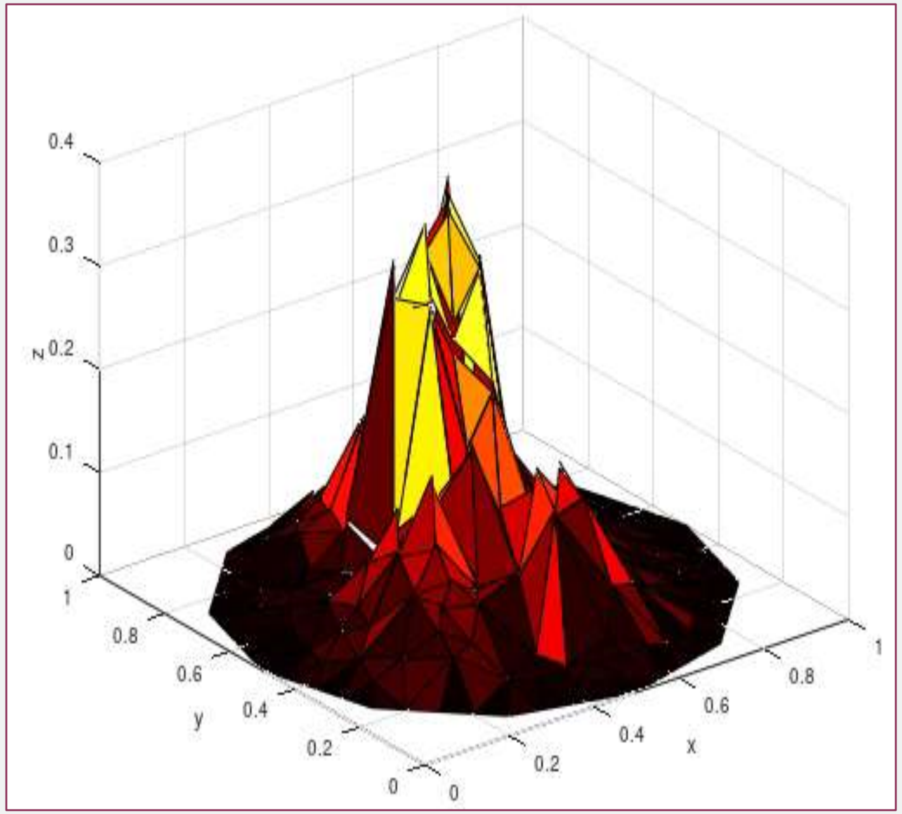
Elliptic Problem Over Mesh Refinement

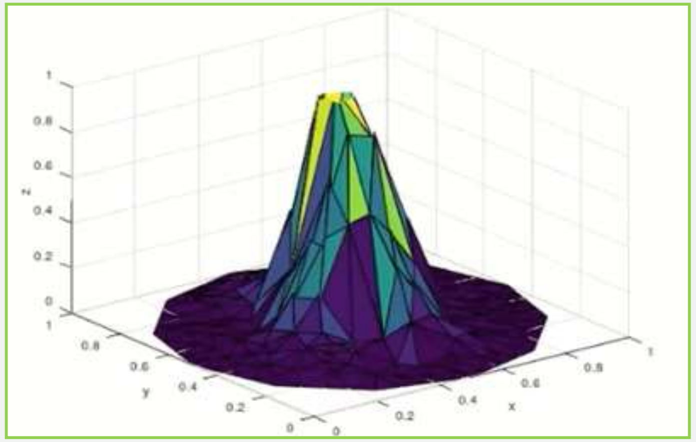
- Keeps shape of exact solution
- Error around interface does not diminish
- Not optimal with Over-penalizing

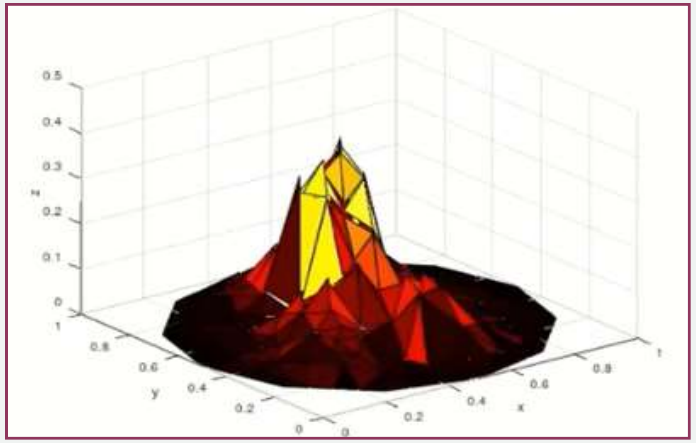


Solving the Interfaced Heat Equation

- Sufficiently small time step
- Oscillation for larger time steps
- Stable in time without over-penalizing







Approximation and Error Plots

Discussion

- Method works for interface conditions equal to zero
- No success with non-zero interface conditions
- Over-penalizing required due to triangulation orientation
- Non-zero interface conditions (WG FEM, DDG FEM)

Acknowledgements

- Dr. Andreas Aristotelous
- Dr. Chuan Li

Questions?