Multi Period Portfolio Optimization via the Hamilton-Jacobi-Bellman Equation

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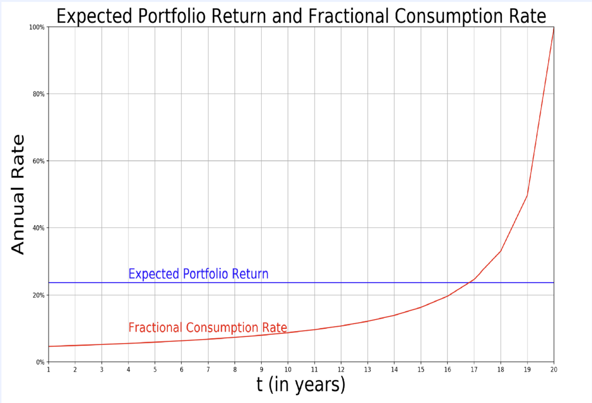
Abstract

In finance, many firms manage portfolios, and a challenge they face is figuring out how to allocate the wealth in the portfolio across risky assets. A mathematical model, called the Hamilton-Jacobi-Bellman (HJB) equation, uses a stochastic control framework describing the evolution of our wealth over a certain time horizon, and provides a framework for finding the optimal investment strategy and maximizing the expected utility of future wealth.
To solve the HJB equation, the continuous time solution is approximated by discretizing the time and using a Monte Carlo simulation to forecast the expected wealth. This model shows how we can combine stochastic control problems by using numerical methods to find the optimal allocation of wealth.

Introduction



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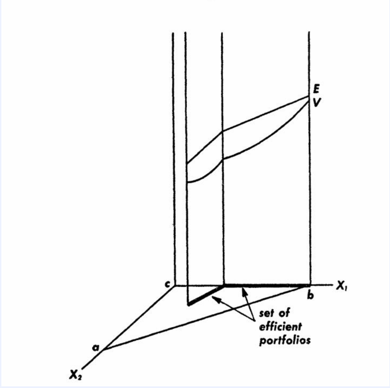
Portfolio Return vs Consumption Rate

Dynamic Portfolio Optimization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Mean Variance Optimization

- HJB equation in finance tells the investor how to allocate assets to maximize "happiness"
- Solves for value function: maximum expected utility of future actions
- Applications: Finance, economics, robotics, energy
- Our focus: Multi-period investment dynamics with stochastic volatility

Model

Bellman Equation

$$V_t(x) = \sup_{\pi_t \in \mathcal{A}_t} \{R_t(x,\pi_t) + \mathbb{E}[\beta V_{t+1}(x_{t+1}^{\pi}) | x_t^{\pi} = x; \pi_t]\}$$

Dynamic Programming Equation

$$V_0 = \sup_{\pi \in \mathcal{A}} \mathbb{E}[\sum \beta^t R_t(X_t,\pi_t) + \beta^N R_N(X_N)]$$

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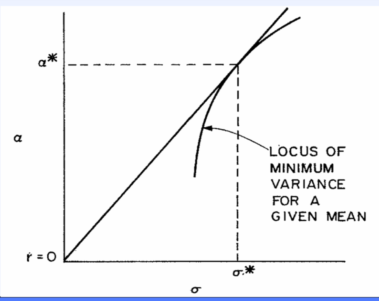

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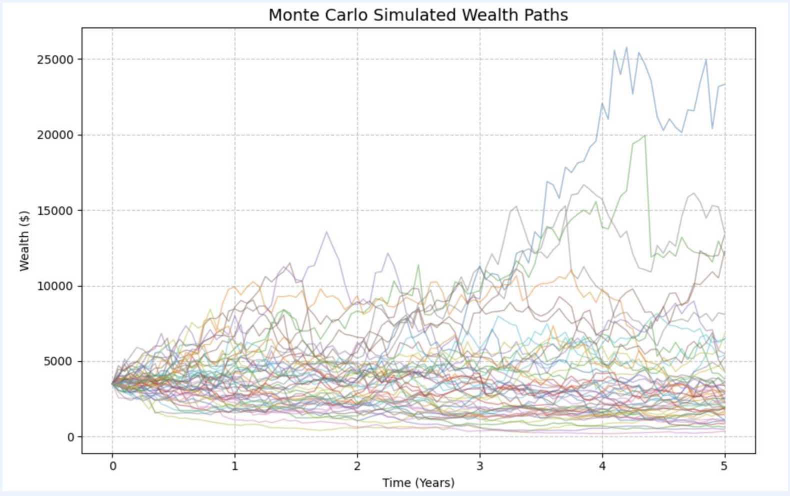
Risk Return tradeoff and Efficient Frontier

Numerical Solution

Expected future portfolio Value



Least Squares Monte Carlo Simulation

- Forward Simulation: Generate wealth paths under random portfolio allocations
- Value Function Approximation: $H^{-1}(\beta V_{t+1}(X_{t+1}^m)) = \Lambda_t^T L(X_t^m, \pi_t^m) + \epsilon_t^m$
- Regression coefficients: $\widehat{\Lambda} = \arg\min_{\Lambda_t} \Sigma[H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \Lambda_t^T L(X_t^m, \pi_t^m)]^2$
- Residuals: $\hat{\epsilon}^m{}_t = H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \widehat{\Lambda}_t^m L(X_t^m, \pi_t^m)$
- Smearing Estimate: $\widehat{H}^B(\widehat{\Lambda}_t^T L(x_t,\pi_t)) = \frac{1}{M} \sum H(\widehat{\Lambda}_t^T L(X_t,\pi_t) + \hat{\epsilon}_t^m)$
- Policy Optimization: $\pi_t^*(X_t^m) = \arg\max_{\pi_t \in \mathcal{A}_t} \{R_t(\widehat{X}_t^m, \pi_t) + \widehat{\Phi}_t(\hat{X}_t^m, \pi_t)\}$
- Backward update: $\widehat{V}_t(X_t) = R_t(X_t, \pi_t^*(X_t)) + \beta \widehat{V}_{t+1}(X_{t+1})$

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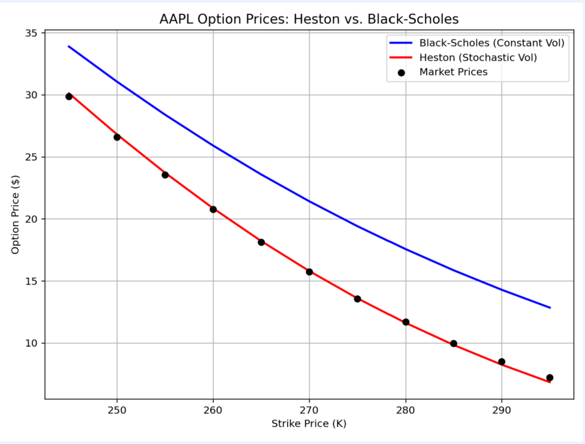

Heston Model vs Black-Scholes

Stochastic Volatility

Heston Model Dynamics:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{1,t}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t}$$

$$dW_{1,t}, dW_{2,t} = \rho dt$$

- dS_t tells us how the underlying asset price moves
- dv_t tells us how the variance (volatility squared) moves

Forecasting Stochastic Volatility:

$$\frac{\widehat{W}}{T} = (v_0 - \theta)\left(\frac{1 - e^{-\kappa T}}{\kappa T}\right) + \theta$$

- $\frac{\widehat{W}}{T}$ tells us the expected variance over time T
- Allows us to use forward looking volatility in portfolio optimizer

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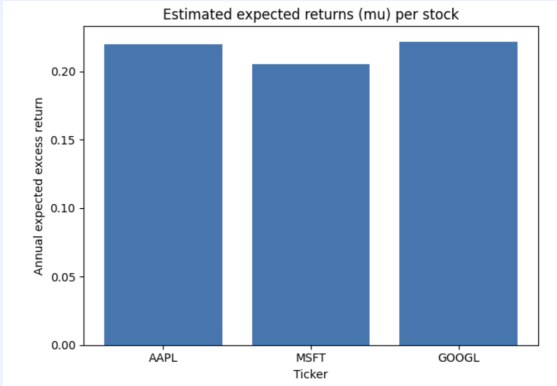
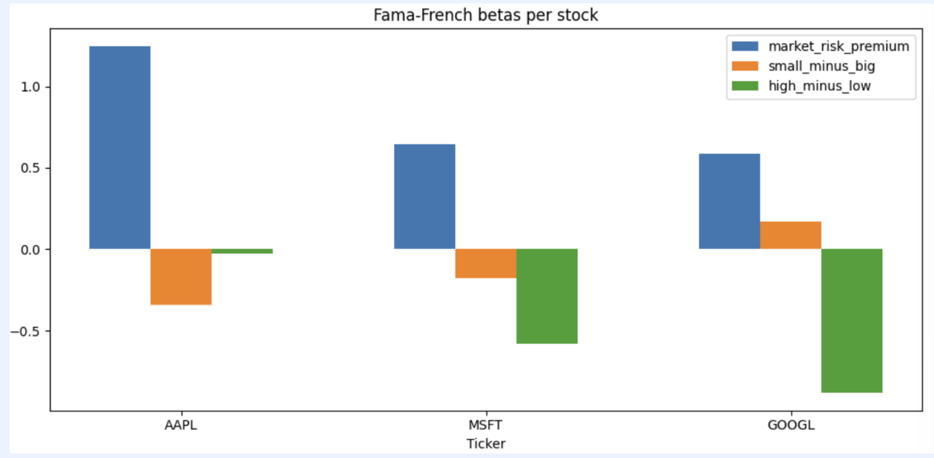
Three Factor Fama-French Factor Model

$$R_i - R_f = \alpha_i + \beta_{i,M}(R_M - R_f) + \beta_{i,SMB} * SMB + \beta_{i,HML} * HML + \epsilon_i$$

- Expected returns are estimated by regressing daily returns using three systematic risk factors
- $R_M - R_f$: Market risk premium
- SMB: captures size effect of small vs large cap
- HML: Captures value vs growth tilt
- Higher β_M: More market exposure
- Higher β_{SMB}: Tilt toward small-cap behavior
- Higher β_{HML}: Tilt toward value stocks [4]

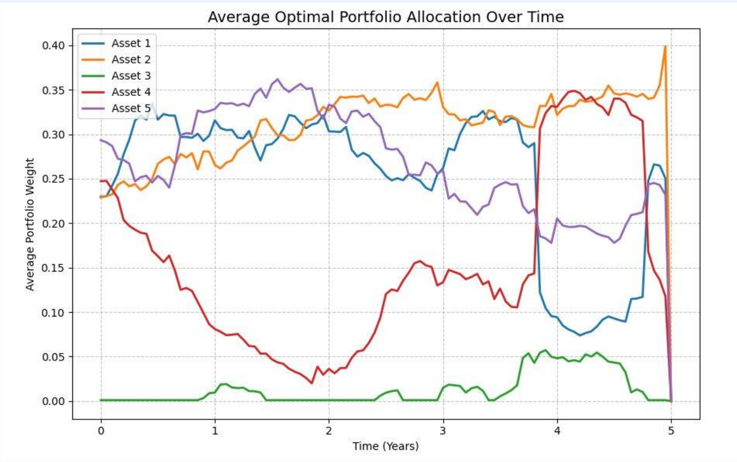


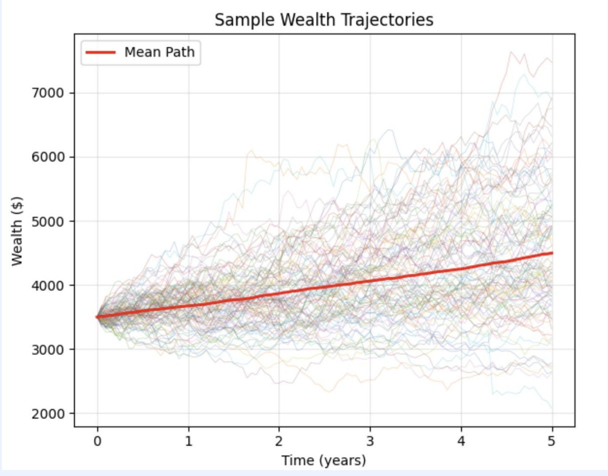
Future Returns for Tech Stocks

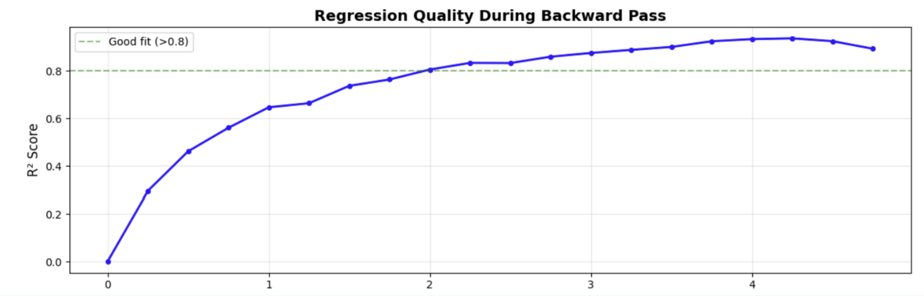


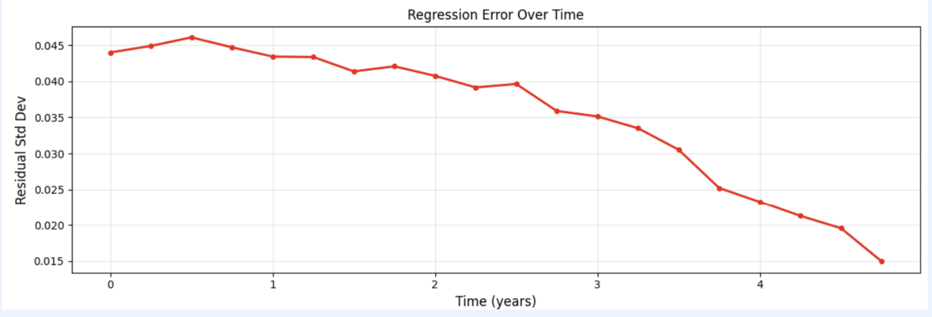
Fama-French Factor Exposure Per Stock

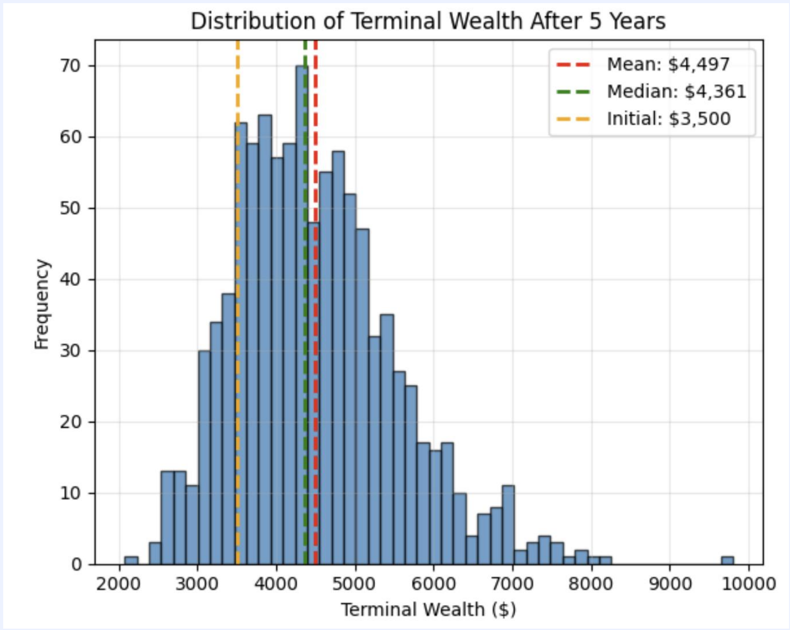
Numerical Experiments

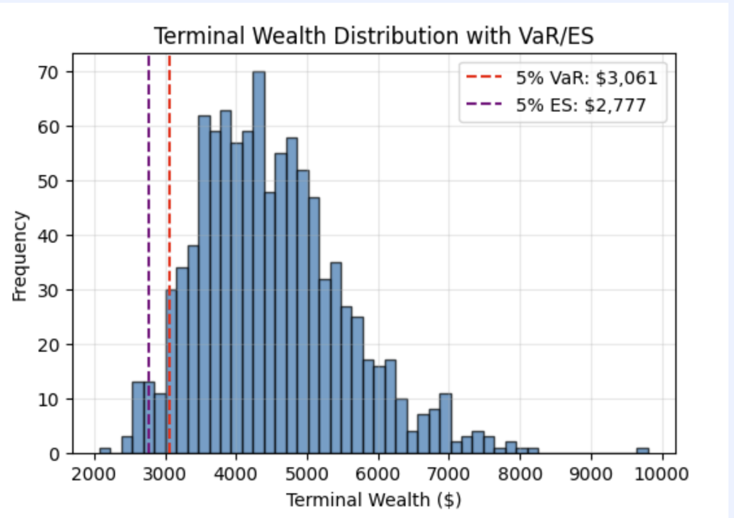


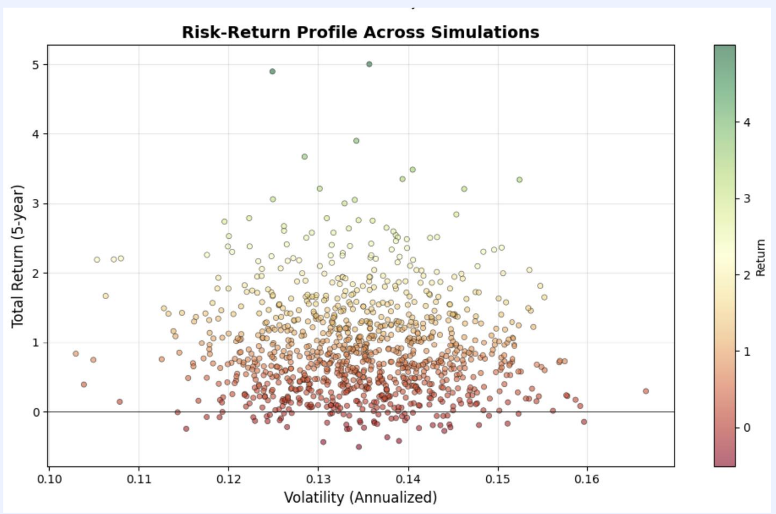


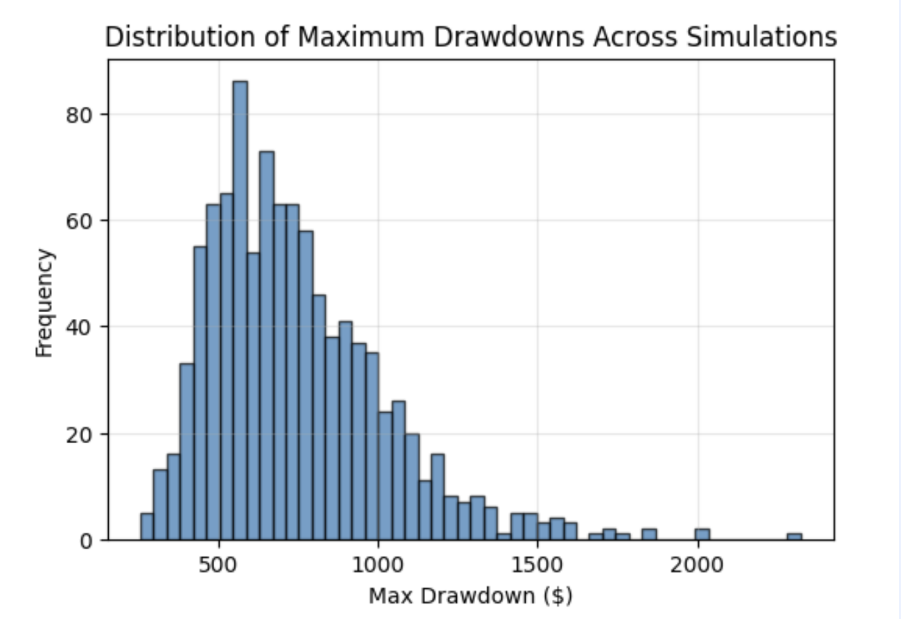


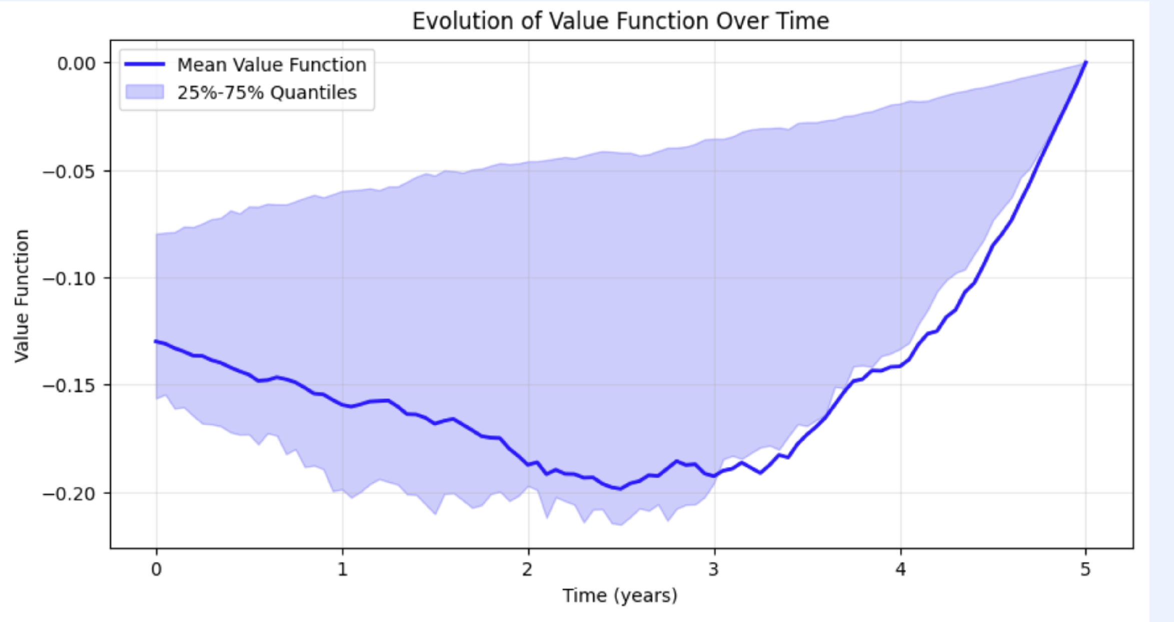












Conclusion and Summary

In conclusion we see that the HJB equation provides us the ability to determine the optimal investment strategy based on the investors risk preference. Using numerical methods, we are able to maximize expected wealth while mitigating potential risk demonstrating the trade-off between growth and loss. The results show how the equation is able to dynamically adjust to the changing market conditions, and how we can incorporate extensions such as stochastic volatility and multi-factor return models. Future work will explore adding constraints to the equation using quadratic programming and building a neural network to better approximate the value function.

References

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