Comparing Methods for Finding Roots of Polynomials

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Motivation

- Finding the roots of a polynomial has been a mathematical problem of interest for hundreds of years.
- For polynomials of degree ≤ 4, algebraic solutions exist to find the exact roots of polynomials.
- However, those algebraic formulas raise a question: Can such formulas produce accurate roots when programmed and executed on a modern computer?

Review

- Recall from high school that for a quadratic of the form $ax^2 + bx + c = 0$, x is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

- However, the Quadratic Formula can be inaccurate for extreme values, particularly for very large values of b.
- Consider the discriminant, defined as

$$\sqrt{b^2 - 4ac} \tag{2}$$

- If a **and** c are small relative to a large value b^2, the discriminant can be approximated as

$$\sqrt{b^2 - 4ac} \approx \sqrt{b^2} = b \tag{3}$$

- Using this approximation, the Quadratic Formula becomes

$$x \approx \frac{-b + b}{2a} = \frac{0}{2a} = 0 \tag{4}$$

Low-Entropy Method

- **Low-Entropy:** An expression that has its terms organized in a way to make their origin evident.
- The Low-Entropy Method is based on the original quadratic formula and aims to resolve truncation issues.
- This method eliminates truncation by subtracting two close values **before** numerical evaluation.
- Middlebrook introduces two terms, Q and F and defines them as follows:

$$Q = \frac{\sqrt{(c/a)}}{b/a} \tag{5}$$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \tag{6}$$

- The two roots of the quadratic are given by:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad x_2 = -\frac{b}{a}F \tag{7}$$

Example of Low-Entropy Method

- Middlebrook used the example of $a = c = 1$ and $b = 45,000$ to show the motivation behind the Low-Entropy Method.
- Using a 10 digit calculator, the smaller root using the traditional formula is $2.000000000 * 10^{-5}$.
- The actual value, to 9 significant digits, is $2.22222222 * 10^{-5}$

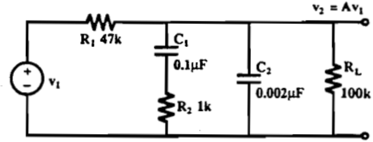

Fig.2. A circuit example to illustrate the improved formula for quadratic roots.

Exact
$\omega_1 = Q\omega_0\frac{1}{F}, \quad \omega_2 = \frac{\omega_0}{Q}F$
$\frac{\omega_1}{\omega_2} = \frac{Q^2}{F^2}$

Bisection Method

- The Bisection Method is a numerical technique for approximating roots of a polynomial.
- The Bisection Method follows these steps:
  1. Choose an interval.
  2. Calculate the midpoint.
  3. Evaluate $f(c)$.
  4. Repeat the process until the interval is sufficiently small.

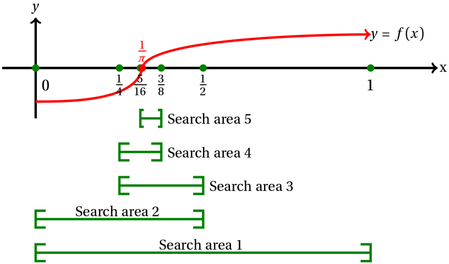

Figure: Visualizing the Bisection Method

Newton's Method

- Newton's method is an iterative numerical technique for finding roots using differential calculus.
- The method follows these steps:
  1. Choose an initial guess close to the actual value.
  2. Compute the next approximation using Newton's iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (8)$$

  3. Repeat the previous step until $|x_{n+1} - x_n|$ is sufficiently small.

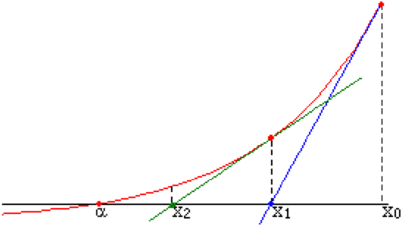

Figure: Newton's Method Visualized

Results for Quadratic Case ($a = c = 1$)

$b = -5 \times 10^{11}$

Method	Small Root	Big Root	Iterations
Quadratic Formula	0.000	5.000e11	N/A
Bisection	2.000e-12	5.001e11	51
Newton's	2.000e-12	5.000e11	3
Low-Entropy	2.000e-12	5.000e11	N/A

$b = -5.3 \times 10^{11}$

Method	Small Root	Big Root	Iterations
Quadratic Formula	0.000	5.300e11	N/A
Bisection	1.887e-12	5.299e11	51
Newton's	1.887e-12	5.300e11	3
Low-Entropy	1.887e-12	5.300e11	N/A

Graphical Representation for Quadratic Results

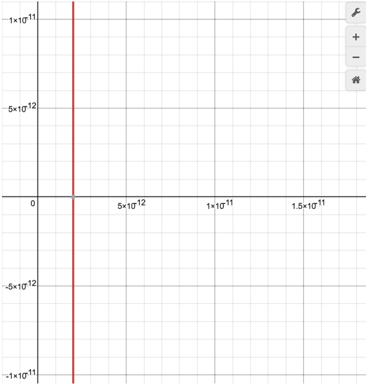

Figure: $b = -5.0 \times 10^{11}$

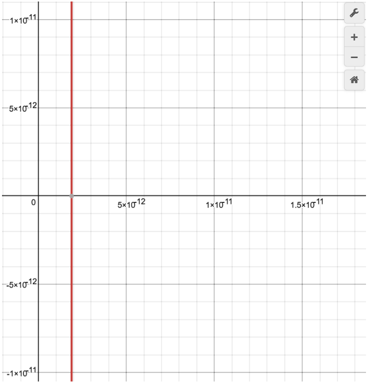

Figure: $b = -5.3 \times 10^{11}$

Cubic Formula

- The solution for a general cubic polynomial, $ax^3 + bx^2 + cx + d = 0$ is given by the following formula:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

- The cubic formula is quite lengthy and involves the subtraction of many terms, leading to truncation errors for extreme coefficient values.

Results for Cubic Case (With $a = c = d = 1$)

$b = -8 \times 10^9$

Method	Root 1	Root 2	Root 3	Iterations
Cubic Formula	8.000e9	No real solution!	No real solution!	N/A
Bisection	8.000e9	1.118e-05	-1.144e-05	51
Newton's	8.000e9	1.118e-05	-1.144e-05	21

$b = -9 \times 10^9$

Method	Root 1	Root 2	Root 3	Iterations
Cubic Formula	9.000e9	25.810	-25.810	N/A
Bisection	9.000e9	1.054-5	-1.049e-5	51
Newton's	9.000e9	1.054e-5	-1.049e-5	21

Graphical Representation for Cubic Results

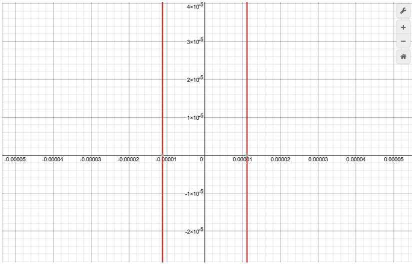

Figure: $b = -8 \times 10^9$

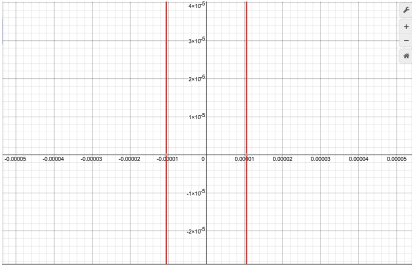

Figure: $b = -9 \times 10^9$

Conclusion

- The standard algebraic formulas work well for most cases when coefficients do not have extreme values.
- Due to rounding errors, algebraic formulas may cause the smaller root(s) to differ significantly from the results obtained numerically.
- Generally speaking, the bisection method is the least efficient method in terms of computation time due to its slow rate of convergence.

Further Research

- While this presentation mentioned second-degree and third-degree polynomials, further investigation is needed for quartic polynomials.
- Further comparisons with extreme values for a and c can also be explored.
- Additionally, further exploration is needed to understand why low-entropy forms may or may not exist for cubic and quartic formulas.

References

- Middlebrook, R.D. (1992). Methods of Design-Oriented Analysis: The Quadratic Equation Revisited. *Proceedings. Twenty-Second Annual conference Frontiers in Education*, 95-102. https://doi.org/10.1109/fie.1992.683365
- Australian Mathematical Sciences Institute. (n.d.). The bisection method. https://amsi.org.au/ESA_Senior_Years/Senior-Topic3/3j/3j_2content_1.html