Mathematical Card Tricks

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Abstract

Amaze and amuse your family and friends armed with just a deck of cards and a little insider knowledge.

Mathematics underpins numerous classic amusements with cards, from forcing to prediction effects, and many such tricks have been written about for general audiences by popularizers such as Martin Gardner.

The mathematics involved ranges from simple "card counting" (basic arithmetic) to parity principles, to surprising shuffling fundamentals (Gilbreath and Faro) discovered in the second half of the last century.

We'll discuss several original and totally different principles discovered since 2000, as well as how to present them as entertainments in ways that leave audiences (even students of mathematics) baffled as to how mathematics could be involved.

Consider a Regular Deck of 52 Playing Cards

Note that $52 = 26 + 26 = 13 + 13 + 13 + 13$.

A deck has 26 black ($\clubsuit$ & $\spadesuit$) and 26 red ($\heartsuit$ & $\diamondsuit$) cards.

There are 13 of each of the 4 suits:

$\clubsuit$ (Clubs), $\heartsuit$ (Hearts), $\spadesuit$ (Spades), $\diamondsuit$ (Diamonds)

A$\clubsuit$ (Ace), 2$\clubsuit$, $\ldots$, 10$\clubsuit$, J$\clubsuit$ (Jack), Q$\clubsuit$ (Queen), K$\clubsuit$ (King),

A$\heartsuit$ (Ace), 2$\heartsuit$, $\ldots$, 10$\heartsuit$, J$\heartsuit$ (Jack), Q$\heartsuit$ (Queen), K$\heartsuit$ (King),

A$\spadesuit$ (Ace), 2$\spadesuit$, $\ldots$, 10$\spadesuit$, J$\spadesuit$ (Jack), Q$\spadesuit$ (Queen), K$\spadesuit$ (King),

A$\diamondsuit$ (Ace), 2$\diamondsuit$, $\ldots$, 10$\diamondsuit$, J$\diamondsuit$ (Jack), Q$\diamondsuit$ (Queen), K$\diamondsuit$ (King)

What if you don't like card tricks?

Spend the next 45 minutes productively. List all possible rearrangements of:

$$A\clubsuit,\ 2\clubsuit,\ \ldots,\ K\clubsuit,$$
$$A\heartsuit,\ 2\heartsuit,\ldots,\ K\heartsuit,$$
$$A\spadesuit,\ 2\spadesuit,\ldots,\ K\spadesuit,$$
$$A\diamondsuit,\ 2\diamondsuit,\ \ldots,\ K\diamondsuit.$$

What if you don't like cards at all?

List all possible rearrangements of the 52 white notes on a piano, using no note twice.

Little Fibs

Shuffle the deck well, and have one card each selected by two spectators.

They remember their cards (value and suit), share the results with each other, and tell you the sum of the chosen card values.

You soon announce what each individual card is!

Secret Number 1:

When several numbers are added up, each one can be determined from the sum.

Secret Number 2:

Totally free choices of cards are offered, but only from a controlled small part of the deck.

The possibilities are narrowed down by having six key cards at the top of the deck at the start, in any order, and keeping them there throughout some fair-looking shuffles.

The mathematics is about numbers (card values).

Secret Number 3:

You've memorized the suits of the top half dozen key cards.

We use the Fibonacci numbers: start with 1, 2; add to get the next one. Repeat.

$$1 + 2 = 3,$$

$$2 + 3 = 5,$$

$$3 + 5 = 8,$$

$$5 + 8 = 13, \text{ and so on.}$$

The list of Fibonacci numbers continues forever:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

For magic with playing cards, we focus on the first six of these, namely the **Little Fibs**, i.e., 1, 2, 3, 5, 8, and 13, agreeing that $1 =$ Ace and $13 =$ King.

Cconsider particular cards with those values, e.g.,

A♣, 2♡, 3♠, 5♢, 8♣, K♡ (CHaSeD order).

If any two cards are selected from these, they can be determined from the sum of their values, because of the *unaddition property*:

Possible totals arise in only one way.

It's easy to break up any sum into the two Fibs it's made up from: just peel off the largest possible Fib, what's left is the other one. E.g.,

$$\begin{aligned} 8 &= 5 + 3, \\ 10 &= 8 + 2, \\ 14 &= 13 + 1, \\ 18 &= 13 + 5. \end{aligned}$$

Other lists of numbers work: the Lucas sequence 2, 1, 3, 4, 7, 11, 18, ..., is a kind of generalized Fibonacci sequence. If we omit the 2, it too has the desired "unaddition" property.

We don't need generalized Fibonacci sequences.

The numbers 1, 2, 4, 6, 10 work.

So do 1, 2, 5, 7, 13.

And?

Can we find a set of five or six interesting numbers such that *all* subsets of any size generate unique sums?

The Three Scoop Miracle

Hand out the deck for shuffling. A spectator is asked to call out her favorite ice-cream flavour; let's suppose she says, "Chocolate."

Take the cards back, and take off about a quarter of the deck. Mix them further until told when to stop.

Deal cards, one for each letter of "chocolate," before dropping the rest on top as a topping.

This spelling/topping routine is repeated twice more—so three times total.

The Three Scoop Miracle

Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavour.

Have the spectator press down hard on the final top card, asking her to magically turn it into a specfic card, say the 4♠. When that card is turned over it is found to be the desired card.

Published 21 Oct 2004 at MAA.org as "Low Down Triple Dealing" as the first *Card Colm*, dedicated to Martin on the occasion of his 90th birthday. See the "Little Fibs" video on *Numberphile*.

Low Down Triple Dealing

The key move here is a *reversed transfer* of a fixed number of cards in a packet—at least half—from top to bottom, done three times total.

The dealing out of k cards from a packet that runs $\{1, 2, \ldots, k-1, k, k+1, k+2, \ldots, n-1, n\}$ from the top down, and then dropping the rest on top as a unit, yields the rearranged packet

$$\{k+1, k+2, \ldots, n-1, n, k, k-1, \ldots, 2, 1\}.$$

True, but it hides what is really going on!

Low Down Triple Dealing

When $k \geq \frac{n}{2}$, doing this three times brings the original bottom card(s) to the top.

Given a flavour of length k and a number $n \leq 2k$, the packet of size n breaks symmetrically into three pieces T, M, B of sizes $n-k$, $2k-n$, $n-k$, such that the count-out-and-transfer operation (of k cards each time) is

$$T, M, B \to B, \overline{M}, \overline{T},$$

where the bar indicates reversal within that piece.

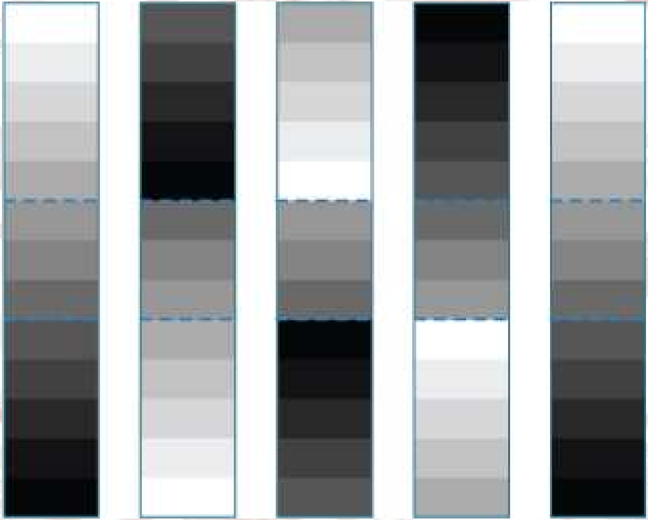
Low Down Triple Dealing Generalized?

Using this approach, the Bottom to Top (with three moves) property can be proved. Actually...

The Bottom to Top Property is only 75% of the story. Here's the real scoop:

> **The Period 4 Principle** *If four reversed transfers of k cards are done to a packet of size n, where $k \geq \frac{n}{2}$, then every card in the packet is returned to its original position.*

Proof without words



Poker Hand Control

Hand out the deck to a spectator for shuffling, the more jumbled it is the better. Deal out two piles of five cards in an alternating way, remarking, "Let's deal out two poker hands. In a moment I'm going to let you decide who gets which cards. I just want to take a peek at what we have here."

Pick up one hand of cards, glance at the faces, then pick up the other hand, and look at those faces, tucking one hand of cards behind the other. Mutter vague (true!) things about what you see, such as, "Interesting, a pair of 4s and a Jack, Queen and King." Then turn the ten card packet face down.

Poker Hand Control

Declare, "As I said, *you* get to decide who gets which cards."

Holding the cards face down, take the top two off and say, "One of these is your hole card, the other is mine. Which do you want, top or bottom?"

Whichever card is claimed, place it face down in front of the spectator, and place the other face down in front of yourself. Say something like, "Did I mention you'd get the pair of 4s? Maybe I shouldn't have said that, I wouldn't want to influence your decisions in any way."

Poker Hand Control

Hold up the next two cards from the packet, and ask, “Which do you want, top or bottom? The other goes underneath.” Whichever card is claimed, add it to the spectator’s face-down pile *and tuck the other one underneath the packet in your hand!*

There has been a subtle change in procedure here, and it is repeated five more times, by which time the spectator may have forgotten the different nature of the first choice offered. Once more, hold up the next two cards from the packet, and ask, “Which do you want, top or bottom? The other goes underneath.”

Poker Hand Control

Act accordingly. The spectator now has three cards in her pile.

Then say, "You also get to pick which cards I get; it couldn't be more fair." Hold up the next two cards from the packet, and ask, "*Which one do I get*, top or bottom? The other goes underneath."

Whichever card is indicated, add it to your pile and tuck the other one underneath the shrinking packet in your hand. Do the same thing one more time.

Poker Hand Control

At this stage, you have four cards in your hand and each pile on the table has three cards in it. Say, "Back to you," as you once more offer the spectator one of the top two cards, placing the other underneath. Then say, "One more for me," as the spectator selects a fourth card for your pile from the top two of the three left in the packet in your hand.

Tuck the other underneath the sole remaining card and casually say, "One final one for each of us," *putting the top one on the spectator's pile and the other one on yours.*

Poker Hand Control

Now have the spectator pick up her five cards and look at their faces, as you do likewise with yours.

Ask, “Did you get those two 4s like I predicted?” Have her display her hand face up, she will indeed have the cards mentioned.

Congratulate her, and add, “Unfortunately, you gave me a pair of 9s, so I guess I win this time!”

Throw your cards down for all to see.

Repeatable Poker Hand Control

Similar results can be obtained with the next ten cards from the deck, and in our experience the spectator is usually happy to give it another try.

If you wish to be nice, you can let the spectator win this time, and even predict that outcome.

It's quite baffling, you seemingly controlling two poker hands with ten random cards, while the spectator makes almost all of the decisions.

Explanation Part I: Birthday Card Matches

Two unrelated concepts make this effect possible. First, **The Birthday Card Match Principle:**

Given ten random cards from a regular 52-card deck, then it's very likely (98%) that there are at least two cards of the same value among them.

This says that most of the time, there will be at least a pair (in the poker sense) somewhere in the two hands first dealt out.

Why?

The inevitability of birthday coincidences

With over 60 randomly selected people, the chances of a birthday match are 99.4%, though you *can* handpick 366 people with different birthdays!

With just 23 randomly selected people, the chances of a birthday match are a little over 50%.

("The Birthday Paradox")

The key to estimating such probabilities is to turn things around, and focus on the chances of there being no match, noting that

$$\mathit{Prob}(\geq \mathit{one\ match}) = 1 - \mathit{Prob}(\mathit{no\ match}).$$

Poker with Ten Random Cards

If k cards are picked at random, then since there are four cards of each value, the chance of getting at least one matching value (i.e., *a pair or better*) is:

$$1 - \frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \ldots \times \frac{52-4k+4}{52-k+1}.$$

For 5 cards, this comes out to be about 50%. For 8 or 9 cards, it's about 89% or 95%, respectively.

For 10 cards, it's about 98%.

Only 2% of the time will you be unlucky!

Explanation Part II: Bill Simon's contribution (1964)

> *It's easy to give the illusion of free choices while really controlling exactly how to split a packet of 8 cards into two piles of 4.*

In fact, you retain control of the division in one key sense: the bottom 4 cards end up in the second pile.

E.g., if you start with 4 red cards of top of 4 blacks, the piles maintain colour separation, with the reds in the first pile and the blacks in the second pile.

For poker purposes, it's winning and losing cards that are controlled, not reds and blacks.

The inevitability of poker hand control

Given 10 random cards, *assuming that there is at least one matching pair*, the poker hand control is based on your making sure that the “winning cards” are in the bottom four, with as little suspicion-arousing rearrangement as possible.

Those are positions seven to ten, from the top.

The “losing cards” are in positions three to six.

The top two cards must not impact which of you wins (i.e., they aren’t “deal-breakers”).

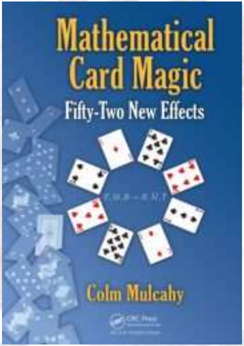
Chapter 10 (Mathematical Card Tricks) of Martin Gardner's "The Scientific American Book of Mathematical Puzzles and Diversions" (originally 1959, later republished as "Hexaflexagons and Other Mathematical Diversions") opens with this cautionary exchange and commentary, before moving on to an expository tour de force.

Somerset Maugham's short story "Mr Know-All" contains the following dialogue:

> *"Do you like card tricks?"*
> *"No, I hate card tricks.""*
> *"Well, I'll just show you this one.""*

After the third trick, the victim finds an excuse to leave the room. His reaction is understandable. Most card magic is a crashing bore unless it is performed by skillful professionals. There are, however, some "self-working" card tricks that are interesting from a mathematical standpoint.

Cool, Colm & Collected



Mathematical Card Magic: Fifty-Two New Effects

AK Peters/CRC Press, 2013. Full colour, 380 pages.

Endorsed by Max Maven, Ron Graham, Persi Diaconis, Art Benjamin & Lennart Green.

13 main chapters, each with 4 effects. Largely original material.

Made that list of rearrangements of 52 items?

Checked it twice? It should have 52! items on it, i.e.,

80658175170943878571660636856403766975289505440883277824000000000000.

That's approximately

$$8 \times 10^{67}$$

This number is larger than the current estimates of the total number of elementary particles in our galaxy.

That's a very big deal indeed.

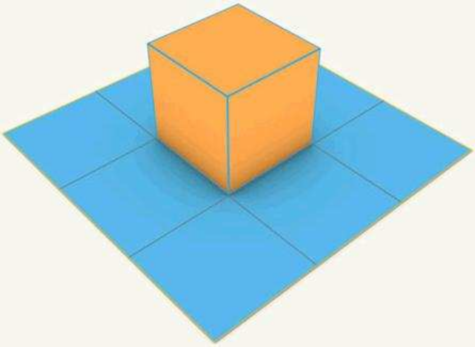
Imagine two seeds released into the air from the top of a windy mountain on different days.

What are the chances that the seeds end up in the exact same place?

It's very very very very small.

But it's much much much much bigger than the probability that two well-shuffled decks of cards are in the same order.

Let's wrap it up (from Cut-the-Knot site)



Wrap the gold cube completely with the blue paper!

All cutting and folding must be along grid lines.

The paper must remain in one piece.