# Generalizations of some $q$-product Identities of Ramanujan and others

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Abstract. By considering certain limiting cases of a WP-Bailey chain discovered by Andrews, and also limiting cases of certain classical summation formulae for basic hypergeometric series, we derive new expressions for certain Lambert series in terms of basic hypergeometric series. In some cases, the resulting series involve an arbitrary Bailey pair. This allows for the derivation of new basic hypergeometric expansions for some $q$-products and series that Ramanujan expressed in terms of Lambert series. Some of Ramanujan's identities are extended to more general relations containing one or more free parameters.

## 1. Introduction

In the present paper certain limiting cases of known identities with one or more free parameters are considered. These limiting cases lead to several different representations for the same function in each case, one of these representation involving Lambert series (see Theorem 3.1, Corollaries 3.4 and 4.1, and Theorems 5.1, 6.1 and 7.1). While the derivation of these limiting cases themselves is elementary, our reason for doing so is that they permit new representations (in terms of basic hypergeometric series) for various functions which also have representations in terms of Lambert series. In some cases, the new representation may involve an arbitrary Bailey- or WP-Bailey pair, thus leading to a separate identity for each such pair.

As an example of this we consider the following identity of Ramanujan, [**4**, Entry 4, Ch. 21], which states that

$$(1.1)\quad 1+6\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-30\sum_{n=1}^{\infty}\frac{nq^{5n}}{1-q^{5n}} = \frac{\left((q;q)_\infty^{12}+22q(q;q)_\infty^6(q^5;q^5)_\infty^6+125q^2(q^5;q^5)_\infty^{12}\right)^{1/2}}{(q;q)_\infty(q^5;q^5)_\infty}.$$

As an implication of one of the results in the present paper, we show that Equation (1.1) can be extended as follows: if $(\alpha_n(1,q),\beta_n(1,q))$ is any Bailey pair

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with respect to 1 (see below for the definition of a Bailey pair), then

$$
\begin{aligned}
(1.2)\quad 1 &+ 6\sum_{n=1}^{\infty}(q,q;q)_{n-1}q^n\beta_n(1,q) - 6\sum_{n=1}^{\infty}\frac{q^n\alpha_n(1,q)}{(1-q^n)^2}\\
&- 30\sum_{n=1}^{\infty}(q^5,q^5;q^5)_{n-1}q^{5n}\beta_n(1,q^5) + 30\sum_{n=1}^{\infty}\frac{q^{5n}\alpha_n(1,q^5)}{(1-q^{5n})^2}\\
&= \frac{\left((q;q)_\infty^{12}+22q(q;q)_\infty^6(q^5;q^5)_\infty^6+125q^2(q^5;q^5)_\infty^{12}\right)^{1/2}}{(q;q)_\infty(q^5;q^5)_\infty}.
\end{aligned}
$$

Ramanujan's original result may then be seen as the special case deriving from the "trivial" Bailey pair ($\alpha_0(1,q)=\beta_0(1,q)=1$, and for $n>0$, $\alpha_n(1,q)=0$ and $\beta_n(1,q)=1/(q,q;q)_n$), after employing the identity

$$
\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.
$$

Any Bailey pair $(\alpha_n(1,q),\beta_n(1,q))$, including some with free parameters, may be substituted in (1.2) to produce specific identities, and we give some examples later.

Other results in the paper are even more general, extending identities of Ramanujan to identities involving an arbitrary WP-Bailey pair (as opposed to an arbitrary Bailey pair) - see, for example, Eq. (3.4) below.

Some of the new identities lack the flexibility of containing an arbitrary Bailey pair or arbitrary WP-Bailey pair, but are still quite interesting. As an example, we have that if $b$ and $\lambda$ be non-zero complex numbers such that $|q|<\max\{1,|b/\lambda|,|\lambda/b|\}$ and we define

$$
g(b,\lambda,q):=\sum_{n=1}^{\infty}\frac{(1-\lambda q^{2n})(q;q)_{n-1}\left(b,\frac{b}{\lambda};q\right)_n\left(\frac{\lambda^2 q}{b};q^2\right)_n\left(\frac{-q\lambda}{b}\right)^n}{(1-\lambda q^n)\left(\frac{q\lambda}{b},\frac{\lambda^2 q}{b},q;q\right)_n(qb;q^2)_n},
$$

then

$$
\begin{aligned}
g(b,\lambda,q)-g\left(\frac{1}{b},\frac{1}{\lambda},q\right)&=\frac{(1-b)\lambda\left(b^2-\lambda^3\right)}{(1-\lambda)\left(b-\lambda^2\right)\left(b^2-\lambda^2\right)}\\
-\frac{\lambda}{b}\frac{\left(b,\frac{q}{b},-\frac{\lambda^2}{b},-\frac{qb}{\lambda^2};q\right)_\infty(q^2,q^2;q^2)_\infty}{\left(\lambda^2,\frac{q^2}{\lambda^2},\frac{\lambda^2}{b^2},\frac{q^2b^2}{\lambda^2};q^2\right)_\infty}&+\frac{b}{\lambda^2}\frac{\left(b,\frac{q^2}{b},\frac{\lambda^2 q}{b},\frac{qb}{\lambda^2},\frac{\lambda^2}{q},\frac{q^3}{\lambda^2},q^2,q^2;q^2\right)_\infty}{\left(\frac{b}{\lambda^2},\frac{\lambda^2q^2}{b},q,q,\frac{b}{q},\frac{q^3}{b},\lambda^2,\frac{q^2}{\lambda^2};q^2\right)_\infty}.
\end{aligned}
$$

Here and throughout, we use the standard $q$-hypergeometric notation

$$
(a;q)_k:=\prod_{n=1}^{k}(1-aq^{n-1})\text{ and }(a_1,\ldots,a_m;q)_k:=(a_1;q)_k\ldots(a_m;q)_k,
$$

valid for $k\in\mathbb{N}\cup\{\infty\}$.

## 2. Background

The work in the present paper continues the work initiated in two previous papers, [**8**] and [**9**], but left aside for several years. We begin by recalling some definitions. A *WP-Bailey pair* was defined by Andrews [**1**] to be a pair of sequences

$(\alpha_n(a,k,q), \beta_n(a,k,q))$ (if the context is clear, we occasionally suppress the dependence on some or all of $a$, $k$ and $q$) satisfying $\alpha_0(a,k,q) = \beta_0(a,k,q) = 1$ and, for $n > 0$,

$$\beta_n(a,k,q) = \sum_{j=0}^{n} \frac{(k/a;q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(aq;q)_{n+j}} \alpha_j(a,k,q). \tag{2.1}$$

If $k = 0$, then the pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a *Bailey pair with respect to* $a$. In the same paper Andrews [**1**] described two constructions whereby new WP-Bailey pairs could be derived from existing pairs. For more on WP-Bailey pairs and chains, see [**2, 7, 12, 13**].

In [**8**] and [**9**], the second author examined certain limiting cases of Andrews' two constructions to derive some general transformation and summation formulae for WP-Bailey pairs and standard Bailey pairs. These identities were applied to derive new expressions for certain functions that can be represented as certain types of Lambert series. We continue these investigations in the present paper. Firstly, we consider two other special cases of Andrews' initial construction that we overlooked in [**8**]. Secondly, we also consider limiting cases of Jackson's ${}_6\phi_5$ summation formula and $q$-analogues of Watson's ${}_3F_2$- and Whipple's ${}_3F_2$ summation formulae. In both cases we derive new general transformations relating certain basic hypergeometric series to various Lambert series. We then use these new transformations to derive new summation formulae for a number of $q$-products that have known expressions in terms of certain Lambert series, as was done in the previous papers [**8**] and [**9**].

## 3. A second limiting case of Andrews first WP-Bailey chain

One of Andrews' [**1**] WP-Bailey chains imply (see Corollary 1 in [**11**], for example) that if $(\alpha_n, \beta_n)$ satisfy (2.1), then subject to suitable convergence conditions,

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, y, z;q)_n}{(\sqrt{k}, -\sqrt{k}, qk/y, qk/z;q)_n} \left(\frac{qa}{yz}\right)^n \beta_n = \frac{(qk, qk/yz, qa/y, qa/z;q)_\infty}{(qk/y, qk/z, qa, qa/yz;q)_\infty} \sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(qa/y, qa/z;q)_n} \left(\frac{qa}{yz}\right)^n \alpha_n. \tag{3.1}$$

In an earlier paper [**8**] the second author investigated the implications of letting $y \to 1$ in (3.1), namely, that if $(\alpha_n, \beta_n)$ is a WP-Bailey pair, then subject to suitable convergence conditions,

$$\sum_{n=1}^{\infty} \frac{(1-kq^{2n})(z;q)_n(q;q)_{n-1}}{(1-k)(qk, qk/z;q)_n} \left(\frac{qa}{z}\right)^n \beta_n - \sum_{n=1}^{\infty} \frac{(z;q)_n(q;q)_{n-1}}{(qa, qa/z;q)_n} \left(\frac{qa}{z}\right)^n \alpha_n = f(a,k,z,q) \tag{3.2}$$

where

$$\begin{aligned} f(a,k,z,q) &= \sum_{n=1}^{\infty} \frac{(1-kq^{2n})(z,k/a;q)_n}{(1-kq^n)(qk/z, qa;q)_n(1-q^n)} \left(\frac{qa}{z}\right)^n \\ &= -\sum_{n=1}^{\infty} \frac{(1-aq^{2n})(z,a/k;q)_n}{(1-aq^n)(qa/z, qk;q)_n(1-q^n)} \left(\frac{qk}{z}\right)^n \end{aligned} \tag{3.3}$$

$$= \sum_{n=1}^{\infty} \frac{kq^n}{1-kq^n} + \sum_{n=1}^{\infty} \frac{q^n a/z}{1-q^n a/z} - \sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{q^n k/z}{1-q^n k/z}.$$

One special case of (3.2) that we omitted in [**8**] was the result of letting $z \to 1$, and we consider that case now.

THEOREM 3.1. *If $(\alpha_n(a,k,q), \beta_n(a,k,q))$ is a WP-Bailey pair, then subject to suitable convergence conditions,*

$$\sum_{n=1}^{\infty} \frac{(1-kq^{2n})(q,q;q)_{n-1}}{(1-k)(qk,qk;q)_n} (qa)^n \beta_n(a,k,q) - \sum_{n=1}^{\infty} \frac{(q,q;q)_{n-1}}{(qa,qa;q)_n} (qa)^n \alpha_n(a,k,q) = f_1(a,k,q),$$

*where*

$$\begin{aligned} f_1(a,k,q) &= \sum_{n=1}^{\infty} \frac{(1-kq^{2n})(k/a;q)_n (q;q)_{n-1}}{(1-kq^n)(qk,qa;q)_n (1-q^n)} (qa)^n \\ &= -\sum_{n=1}^{\infty} \frac{(1-aq^{2n})(a/k;q)_n (q;q)_{n-1}}{(1-aq^n)(qa,qk;q)_n (1-q^n)} (qk)^n \\ &= \sum_{n=1}^{\infty} \frac{q^n a}{(1-q^n a)^2} - \sum_{n=1}^{\infty} \frac{q^n k}{(1-q^n k)^2}. \end{aligned}$$

PROOF. Divide the identity at (3.2) through by $1-z$ and then let $z \to 1$. The next-to-last equality involving Lambert series follows from the fact that if we define

$$G(z) := \sum_{n=1}^{\infty} \frac{kq^n}{1-kq^n} + \sum_{n=1}^{\infty} \frac{q^n a/z}{1-q^n a/z} - \sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{q^n k/z}{1-q^n k/z},$$

then $G(1) = 0$ and

$$\begin{aligned} &\lim_{z\to 1} \frac{1}{1-z} \left( \sum_{n=1}^{\infty} \frac{kq^n}{1-kq^n} + \sum_{n=1}^{\infty} \frac{q^n a/z}{1-q^n a/z} - \sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{q^n k/z}{1-q^n k/z} \right) \\ &\qquad = \lim_{z\to 1} \frac{G(z)-G(1)}{1-z} = -G'(1) = \sum_{n=1}^{\infty} \frac{q^n a}{(1-q^n a)^2} - \sum_{n=1}^{\infty} \frac{q^n k}{(1-q^n k)^2}. \end{aligned}$$

□

One reason we consider this special case is that certain $q$-products may be expressed in terms of Lambert series of the type exhibited above, and the corollary now permits new representations of such products in terms of basic hypergeometric series to be given – in fact, one such representation for each WP-Bailey pair. We give one example, which leads to a large number of new representations for the product $q(q^5;q^5)_\infty^5/(q;q)_\infty$.

COROLLARY 3.2. Let $(\alpha_n(a,k,q), \beta_n(a,k,q))$ be a WP-Bailey pair. Then, subject to suitable convergence conditions,

$$(3.4)\quad q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty} \frac{(1-q^{10n-2})(q^5,q^5;q^5)_{n-1}q^n}{(1-1/q^2)(q^3,q^3;q^5)_n}\beta_n\left(\frac{1}{q^4},\frac{1}{q^2},q^5\right) + \sum_{n=1}^{\infty} \frac{(1-q^{10n-3})(q^5,q^5;q^5)_{n-1}q^{4n}}{(1-1/q^3)(q^2,q^2;q^5)_n}\beta_n\left(\frac{1}{q},\frac{1}{q^3},q^5\right) - \sum_{n=1}^{\infty} \frac{(q^5,q^5;q^5)_{n-1}q^n}{(q,q;q^5)_n}\alpha_n\left(\frac{1}{q^4},\frac{1}{q^2},q^5\right) - \sum_{n=1}^{\infty} \frac{(q^5,q^5;q^5)_{n-1}q^{4n}}{(q^4,q^4;q^5)_n}\alpha_n\left(\frac{1}{q},\frac{1}{q^3},q^5\right).$$

$$(3.5)\quad q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty} \frac{(1-q^{10n-2})(q^5;q^5)_{n-1}\left(\frac{1}{q};q^5\right)_n q^n}{(1-1/q^2)(1-q^{5n})\left(q^3,q^3;q^5\right)_n} + \sum_{n=1}^{\infty} \frac{(1-q^{10n-3})(q^5;q^5)_{n-1}\left(\frac{1}{q^9};q^5\right)_n q^{4n}}{(1-1/q^3)(1-q^{5n})\left(q^2,q^2;q^5\right)_n} - \sum_{n=1}^{\infty} \frac{(1-q^{10n-4})(q^5;q^5)_{n-1}\left(\frac{1}{q^3};q^5\right)_n\left(\frac{1}{q};q^5\right)_{2n} q^{3n}}{(1-q^{5n-4})(1-q^{5n})\left(q,q^4;q^5\right)_n\left(\frac{1}{q^2};q^5\right)_{2n}} - \sum_{n=1}^{\infty} \frac{(1-q^{10n-1})(q^5;q^5)_{n-1}\left(\frac{1}{q^7};q^5\right)_n\left(q^6;q^5\right)_{2n} q^{2n}}{(1-q^{5n-1})(1-q^{5n})\left(q^4,q^{11};q^5\right)_n\left(\frac{1}{q^3};q^5\right)_{2n}}.$$

PROOF. By [**3**, Entry 18.2.23, Ch. 18]

$$(3.6)\quad q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2},$$

where $(n/5)$ is the Legendre symbol. In Theorem 3.1, replace $q$ with $q^5$ and set $(a,k)=(1/q^4,1/q^2)$ and $(a,k)=(1/q,1/q^3)$ respectively. Add the resulting identities and it can be seen from the last representation of $f_1(a,k,q)$ that

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = f_1(1/q^4,1/q^2,q^5)+f_1(1/q,1/q^3,q^5)$$

and (3.4) now follows. Inserting the WP-Bailey pair (see, for example, [**2**, Eqs. (3.3) and (3.4)])

$$\alpha_n'(a,k) = \frac{1-aq^{2n}}{1-a}\frac{(a,k/aq;q)_n(qa^2/k;q)_{2n}}{(q^2a^2/k,q;q)_n(k;q)_{2n}}\left(\frac{k}{a}\right)^n,$$
$$\beta_n'(a,k) = \frac{(k^2/qa^2;q)_n}{(q;q)_n}.$$

into (3.4) gives (3.5). □

The substitution $a=1$ and $k=e^{\pm i\theta}$ in Theorem 3.1 results in a WP-Bailey pair iteration for a fundamental building block of elliptic modular functions.

Corollary 3.3. Let $\wp(\theta)$ denote the Weierstrass $\wp$-function, normalized so that [**14**, Ex. 35, p. 460]

$$\wp(\theta) = \frac{1}{4}\csc^2\frac{\theta}{2} - \frac{1}{12} + 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 2\sum_{n=1}^{\infty}\frac{nq^n\cos(n\theta)}{1-q^n}.$$

Suppose $(\alpha_n(a,k,q), \beta_n(a,k,q))$ is a WP-Bailey pair. Then, subject to suitable convergence conditions,

$$\begin{aligned}\wp(\theta) = \frac{1}{4}\csc^2\frac{\theta}{2} - \frac{1}{12} &+ \sum_{n=1}^{\infty}\frac{(1-q^{2n}e^{i\theta})(q;q)_{n-1}^2}{(1-e^{i\theta})(qe^{i\theta};q)_n^2}q^n\beta_n(1,e^{i\theta},q)\\ &+ \sum_{n=1}^{\infty}\frac{(1-q^{2n}e^{-i\theta})(q;q)_{n-1}^2}{(1-e^{-i\theta})(qe^{-i\theta};q)_n^2}q^n\beta_n(1,e^{-i\theta},q)\\ &- \sum_{n=1}^{\infty}\frac{q^n\alpha_n(1,e^{i\theta},q)}{(1-q^n)^2} - \sum_{n=1}^{\infty}\frac{q^n\alpha_n(1,e^{-i\theta},q)}{(1-q^n)^2}.\end{aligned}$$

We next consider another special case of (3.2), which follows from setting $k = -a$. This special case also has some interesting applications. We also consider this special case ($k = -a$) of Theorem 3.1. The proofs are straightforward, so are omitted.

Corollary 3.4. If $(\alpha_n(a,k), \beta_n(a,k)) = (\alpha_n(a,k,q), \beta_n(a,k,q))$ is a WP-Bailey pair, then subject to suitable convergence conditions,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1+aq^{2n})(z;q)_n(q;q)_{n-1}}{(1+a)(-qa,-qa/z;q)_n}\left(\frac{qa}{z}\right)^n\beta_n(a,-a)&\\ -\sum_{n=1}^{\infty}\frac{(z;q)_n(q;q)_{n-1}}{(qa,qa/z;q)_n}\left(\frac{qa}{z}\right)^n\alpha_n(a,-a) &= f_2(a,z,q)\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}f_2(a,z,q) &= \sum_{n=1}^{\infty}\frac{(1+aq^{2n})(z,-1;q)_n}{(1+aq^n)(-qa/z,qa;q)_n(1-q^n)}\left(\frac{qa}{z}\right)^n\\ &= -\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(z,-1;q)_n}{(1-aq^n)(qa/z,-qa;q)_n(1-q^n)}\left(\frac{-qa}{z}\right)^n\\ &= 2\sum_{n=1}^{\infty}\frac{q^na/z}{1-q^{2n}a^2/z^2} - 2\sum_{n=1}^{\infty}\frac{aq^n}{1-a^2q^{2n}},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1+aq^{2n})(q,q;q)_{n-1}}{(1+a)(-qa,-qa;q)_n}(qa)^n\beta_n(a,-a)&\\ -\sum_{n=1}^{\infty}\frac{(q,q;q)_{n-1}}{(qa,qa;q)_n}(qa)^n\alpha_n(a,-a) &= f_3(a,q)\end{aligned}$$

where

$$f_3(a,q) = \sum_{n=1}^{\infty}\frac{(1+aq^{2n})(-1;q)_n(q;q)_{n-1}(qa)^n}{(1+aq^n)(-qa,qa;q)_n(1-q^n)}$$

$$= -\sum_{n=1}^{\infty} \frac{(1-aq^{2n})(-1;q)_n (q;q)_{n-1} (-qa)^n}{(1-aq^n)(qa,-qa;q)_n (1-q^n)}$$
$$= 2\sum_{n=1}^{\infty} \frac{q^n a(1+a^2q^{2n})}{(1-q^{2n}a^2)^2}.$$

To give an application of (3.7), we recall another identity of Ramanujan (see [**4**, Entry 3, Ch. 19]):

$$q\psi(q^2)\psi(q^6) = \sum_{n=1}^{\infty} \frac{q^{6n-5}}{1-q^{12n-10}} - \sum_{n=1}^{\infty} \frac{q^{6n-1}}{1-q^{12n-2}},$$

where $\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2;q^2)_\infty(-q;q)_\infty$ (this function also appears in other identities below). Now in (3.7), replace $q$ with $q^6$, set $a = 1/q$ and $z = q^4$, and divide through by 2. Next insert the unit WP-Bailey pair (see [**10**, Eq. (12.15)], for example),

$$\alpha_n(a,k) = \frac{(1-aq^{2n})(a,\frac{a}{k};q)_n}{(1-a)(q,kq;q)_n}\left(\frac{k}{a}\right)^n, \qquad \beta_n(a,k) = \delta_{n,0},$$

in the form $(\alpha_n(1/q,-1/q,q^6), \beta_n(1/q,-1/q,q^6))$ and the following new identity results.

Corollary 3.5. If $|q| < 1$, then

$$q\psi(q^2)\psi(q^6) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{(1-q^{12n-1})\left(q^4,-1;q^6\right)_n (-q)^n}{(1-q^{6n-1})(1-q^{6n})\left(-q^5,q;q^6\right)_n}.$$

## 4. Identities deriving from standard Bailey pairs

Upon letting $k \to 0$ in Theorem 3.1, we get the following corollary, which implies several interesting representations for the Lambert series

$$\sum_{n=1}^{\infty} \frac{aq^n}{(1-aq^n)^2}.$$

Corollary 4.1. If $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair, then subject to suitable convergence conditions,

$$\sum_{n=1}^{\infty}(q,q;q)_{n-1}(qa)^n\beta_n(a,q) - \sum_{n=1}^{\infty}\frac{(q,q;q)_{n-1}}{(qa,qa;q)_n}(qa)^n\alpha_n(a,q) = f_4(a,q), \tag{4.1}$$

where

$$f_4(a,q) = \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}}{(qa;q)_n(1-q^n)}(qa)^n \tag{4.2}$$
$$= -\sum_{n=1}^{\infty} \frac{(1-aq^{2n})(q;q)_{n-1}}{(1-aq^n)(qa;q)_n(1-q^n)}(-a)^n q^{n(n+1)/2}$$
$$= \sum_{n=1}^{\infty} \frac{q^n a}{(1-q^n a)^2}.$$

Replacing $q$ with $q^5$ in (4.1), setting $a = q^{-j}$, for $1 \le j \le 4$, and employing (3.6) in combination with (4.2) gives the following identities.

COROLLARY 4.2. If $|q| < 1$, then

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{j=1}^{4}\left(\frac{j}{5}\right)\sum_{n=1}^{\infty}\frac{(q^5;q^5)_{n-1}q^{(5-j)n}}{(q^{5-j};q^5)_n(1-q^{5n})}$$
$$= -\sum_{j=1}^{4}\left(\frac{j}{5}\right)\sum_{n=1}^{\infty}\frac{(1-q^{10n-j})(q^5;q^5)_{n-1}(-1)^n q^{(5n^2+5n)/2-jn}}{(1-q^{5n-j})(1-q^{5n})(q^{5-j};q^5)_n}.$$

REMARK 4.3. The identity (1.2) is a direct consequence of using (1.1) in conjunction with Corollary 4.1.

## 5. A limiting case of Jackson's ${}_6\phi_5$ summation formula

We now consider limiting cases of a number of other summation formulae from the literature. The applications will not be so general, as the resulting identities do not contain WP-Bailey pairs or standard Bailey pairs, but we will exhibit some interesting consequences.

We first recall Jackson's summation formula for a very-well-poised ${}_6\phi_5$ series [**6**, p. 356, Eq. (II. 20)]:

$$\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,b,c,d;q)_n}{(1-a)\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},q;q\right)_n}\left(\frac{aq}{bcd}\right)^n = \frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}. \tag{5.1}$$

THEOREM 5.1. *Let $a$ be a complex number such that $aq^n \neq 1$ for $n \geq 1$. Then*

$$\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(q,q;q)_{n-1}(qa)^n}{(1-aq^n)(1-q^n)(qa,qa;q)_n} = \sum_{n=1}^{\infty}\frac{aq^n(1+aq^n)}{(1-aq^n)^3} = \sum_{n=1}^{\infty}\frac{n^2a^nq^n}{1-q^n}. \tag{5.2}$$

PROOF. Rewrite (5.1) as

$$\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(a;q)_n(bq,cq,dq;q)_{n-1}}{(1-a)\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},q;q\right)_n}\left(\frac{aq}{bcd}\right)^n$$
$$= \frac{1}{(1-b)(1-c)(1-d)}\left(\frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty} - 1\right).$$

Let $b, c, d \to 1$ and compute the limits on the right side as derivatives. The last equality follows from expanding the $1/(1-q^n)$ terms in the last expression as geometric series, changing the order of summation, and employing a standard summation identity. □

We give one example of an application of (5.2). First, we recall two cubic theta functions from [**5**]:

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \qquad b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2},$$

where $\omega = \exp(2\pi i/3)$.

Corollary 5.2. If $|q| < 1$, then

$$a(q)^3 - b(q)^3 = 27\bigg( \sum_{n=1}^{\infty} \frac{(1-q^{6n-2})(q^3,q^3;q^3)_{n-1}q^n}{(1-q^{3n-2})(1-q^{3n})(q,q;q^3)_n} - \sum_{n=1}^{\infty} \frac{(1-q^{6n-1})(q^3,q^3;q^3)_{n-1}q^{2n}}{(1-q^{3n-1})(1-q^{3n})(q^2,q^2;q^3)_n}\bigg).$$

Proof. By (18.2.10) and (18.2.12) in Chapter 18 of [**3**],

$$a(q)^3 - b(q)^3 = 27\left(\sum_{n=1}^{\infty} \frac{n^2q^n}{1-q^{3n}} - \sum_{n=1}^{\infty} \frac{n^2q^{2n}}{1-q^{3n}}\right).$$

Now use (5.2), with $q$ replaced with $q^3$ and $a = 1/q^2$ and $a = 1/q$, respectively. □

## 6. A limiting case of a $q$-analogue of Whipple's ${}_3F_2$ summation formula

We next recall Bailey's $q$-analogue of Whipple's ${}_3F_2$ sum (see [**6**, Formula II.18, page 355], with $C$ replaced with $-C$ and $d$ replaced with $-d$):

$$ {}_8\phi_7 \left[\begin{matrix} C,\ q\sqrt{C},\ -q\sqrt{C},\ a,\ q/a,\ -C,\ d,\ q/d \\ \sqrt{C},\ -\sqrt{C},\ Cq/a,\ aC,\ -q,\ Cq/d,\ Cd \end{matrix}; q, -C\right] = \frac{(C,Cq;q)_\infty (aCd, aCq/d, Cdq/a, Cq^2/ad; q^2)_\infty}{(Cd, Cq/d, aC, Cq/a; q)_\infty}.$$

If this identity is rewritten as

$$\sum_{n=1}^{\infty} \frac{(1-Cq^{2n})(aq,dq;q)_{n-1}(C,-C,q/a,q/d;q)_n(-C)^n}{(1-C)(Cq/a,aC,Cq/d,Cd,-q,q;q)_n} = \frac{1}{(1-a)(1-d)}\left(\frac{(C,Cq;q)_\infty (aCd, aCq/d, Cdq/a, Cq^2/ad; q^2)_\infty}{(Cd, Cq/d, aC, Cq/a; q)_\infty} - 1\right)$$

and we let, in turn, $a \to 1$ and $d \to 1$ the identities in the next theorem result. We omit the proofs, as they are similar to earlier proofs.

Theorem 6.1. *For $|C| < 1$ and non-zero $d \neq q^n$ for $n \geq 1$,*

$$\sum_{n=1}^{\infty} \frac{(1-Cq^{2n})(q;q)_{n-1}(-C,d,q/d;q)_n(-C)^n}{(1-C)(Cq,Cq/d,Cd,-q;q)_n} = \frac{Cd}{1-Cd} - \frac{C}{1-C} + \sum_{n=1}^{\infty}\left(\frac{(-1)^nq^nCd}{1-q^nCd} - \frac{(-1)^nq^nC/d}{1-q^nC/d}\right), \tag{6.1}$$

*and*

$$\sum_{n=1}^{\infty} \frac{(1-Cq^{2n})(q,q;q)_{n-1}(-C,q;q)_n(-C)^n}{(1-C)(Cq,Cq,C,-q;q)_n} = -\frac{C}{(1-C)^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^nq^nC}{(1-q^nC)^2}.$$

Before giving an example, recall once again that $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$.

Corollary 6.2. If $|q|<1$, then

$$2q\psi(q^4)^2 = \frac{q}{1-q} - \frac{iq}{1-iq} - \sum_{n=1}^{\infty} \frac{(1-iq^{4n+1})(q^2;q^2)_{n-1}(i;q^2)_n(-iq;q)_{2n}(-iq)^n}{(q^3,-q,-q^2;q^2)_n(iq;q^2)_{n+1}}.$$

Proof. Replace $q$ with $q^2$ in (6.1) and then set $C=iq$ and $d=i$. The result follows after some elementary manipulations, upon recalling the following identity of Ramanujan (see [**3**, Entry 18.2.4, P. 397]):

$$\psi^2(q^4) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}}{1-q^{4n+2}}.$$

□

## 7. A limiting case of a $q$-analogue of Watson's ${}_3F_2$ summation formula

The formula in question is the following identity of Bailey (see [**6**, Formula II.16, page 355]):

$${}_8\phi_7\left[\begin{matrix}\lambda,\, q\sqrt{\lambda},\, -q\sqrt{\lambda},\, a,\, b,\, \lambda\sqrt{q/ab},\, -\lambda\sqrt{q/ab},\, ab/\lambda \\ \sqrt{\lambda},\, -\sqrt{\lambda},\, \lambda q/a,\, \lambda q/b,\, \lambda^2 q/ab,\, \sqrt{qab},\, -\sqrt{qab}\end{matrix}; q, -\frac{q\lambda}{ab}\right] = \frac{(\lambda q, \lambda q/ab;q)_\infty}{(\lambda q/a, \lambda q/b;q)_\infty}\frac{(aq,bq,q^2\lambda^2/a^2b, q^2\lambda^2/ab^2;q^2)_\infty}{(q,abq,q^2\lambda^2/ab,q^2\lambda^2/a^2b^2;q^2)_\infty}.$$

The same kind of manipulations as in the proof of Theorem 6.1 leads to the following identities.

Theorem 7.1. *If $b$ and $\lambda$ are non-zero complex numbers such that $|q|<\max\{1,|b/\lambda|\}$ and none of the denominators below vanish, then*

$$\sum_{n=1}^{\infty} \frac{(1-\lambda q^{2n})(q;q)_{n-1}(b,b/\lambda;q)_n(\lambda^2 q/b;q^2)_n(-q\lambda/b)^n}{(1-\lambda q^n)(q\lambda/b,\lambda^2 q/b,q;q)_n(qb;q^2)_n} = \sum_{n=1}^{\infty}\left(\frac{\lambda q^n}{1-\lambda q^n} - \frac{\frac{\lambda q^n}{b}}{1-\frac{\lambda^2 q^{2n}}{b^2}} - \frac{\frac{\lambda^2 q^{2n}}{b}}{1-\frac{\lambda^2 q^{2n}}{b}} + \frac{q^{2n-1}}{1-q^{2n-1}} - \frac{bq^{2n-1}}{1-bq^{2n-1}}\right); \tag{7.1}$$

*If $|q|<\max\{1,1/|\lambda|\}$ and none of the denominators below vanish, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1-\lambda q^{2n})(q,q;q)_{n-1}(1/\lambda;q)_n(\lambda^2 q;q^2)_n(-q\lambda)^n}{(1-\lambda q^n)(q\lambda,\lambda^2 q,q;q)_n(q;q^2)_n} &= \sum_{n=1}^{\infty}\left(-\frac{\lambda q^n}{(1-\lambda q^n)^2} + \frac{\lambda^2 q^{2n}}{(1-\lambda^2 q^{2n})^2} + \frac{q^{2n-1}}{(1-q^{2n-1})^2}\right) \\ &= \sum_{n=1}^{\infty}\left(-\frac{n\lambda^n q^n}{1-q^n} + \frac{n\lambda^{2n}q^{2n}}{1-q^{2n}} + \frac{nq^n}{1-q^{2n}}\right).\end{aligned} \tag{7.2}$$

Proof. Shift the initial "1" from the left side to the right side, divide through by $1-a$ and let $a\to 1$ to get (7.1), then divide through by $1-b$ and let $b\to 1$ to get (7.2). □

We give two applications of this theorem. Before coming to the first of these, we recall a result from [**8**], namely that if $f(a,k,z,q)$ is as defined at (3.3), then

$$f(a,k,z,q) - f\left(\frac{1}{a},\frac{1}{k},\frac{1}{z},q\right) = \frac{(a-k)(1-1/z)(1-ak/z)}{(1-a)(1-k)(1-a/z)(1-k/z)} + \frac{z}{k}\frac{(z,q/z,k/a,qa/k,ak/z,qz/ak,q,q;q)_\infty}{(z/k,qk/z,z/a,qa/z,a,q/a,k,q/k;q)_\infty}.$$

We note two special cases.

$$f(\lambda/b,\lambda,-1,q) - f(b/\lambda,1/\lambda,-1,q) = \frac{(\lambda/b-\lambda)(1+\lambda^2/b)}{(1-\lambda^2/b^2)(1-\lambda^2)} - \frac{\lambda}{b}\frac{(b,q/b,-\lambda^2/b,-qb/\lambda^2;q)_\infty(q^2,q^2;q^2)_\infty}{(\lambda^2,q^2/\lambda^2,\lambda^2/b^2,q^2b^2/\lambda^2;q^2)_\infty}. \tag{7.3}$$

$$f\left(\frac{b}{q},\lambda^2,b,q^2\right) - f\left(\frac{q}{b},\frac{1}{\lambda^2},\frac{1}{b},q^2\right) = \frac{(b/q-\lambda^2)(1-1/b)(1-\lambda^2/q)}{(1-b/q)(1-\lambda^2)(1-1/q)(1-\lambda^2/b)} + \frac{b}{\lambda^2}\frac{(b,q^2/b,\lambda^2q/b,qb/\lambda^2,\lambda^2/q,q^3/\lambda^2,q^2,q^2;q^2)_\infty}{(b/\lambda^2,\lambda^2q^2/b,q,q,b/q,q^3/b,\lambda^2,q^2/\lambda^2;q^2)_\infty}. \tag{7.4}$$

The first consequence of Theorem 7.1 is a somewhat curious reciprocity-type result.

Corollary 7.2. Let $b$ and $\lambda$ be non-zero complex numbers such that $|q| < \max\{1,|b/\lambda|,|\lambda/b|\}$ and define

$$g(b,\lambda,q) := \sum_{n=1}^{\infty}\frac{(1-\lambda q^{2n})(q;q)_{n-1}\left(b,\frac{b}{\lambda};q\right)_n\left(\frac{\lambda^2 q}{b};q^2\right)_n\left(\frac{-q\lambda}{b}\right)^n}{(1-\lambda q^n)\left(\frac{q\lambda}{b},\frac{\lambda^2 q}{b},q;q\right)_n(qb;q^2)_n}. \tag{7.5}$$

If none of the denominators below vanish, then

$$g(b,\lambda,q) - g\left(\frac{1}{b},\frac{1}{\lambda},q\right) = \frac{(1-b)\lambda\left(b^2-\lambda^3\right)}{(1-\lambda)\left(b-\lambda^2\right)\left(b^2-\lambda^2\right)} - \frac{\lambda}{b}\frac{\left(b,\frac{q}{b},-\frac{\lambda^2}{b},-\frac{qb}{\lambda^2};q\right)_\infty(q^2,q^2;q^2)_\infty}{\left(\lambda^2,\frac{q^2}{\lambda^2},\frac{\lambda^2}{b^2},\frac{q^2b^2}{\lambda^2};q^2\right)_\infty} + \frac{b}{\lambda^2}\frac{\left(b,\frac{q^2}{b},\frac{\lambda^2 q}{b},\frac{qb}{\lambda^2},\frac{\lambda^2}{q},\frac{q^3}{\lambda^2},q^2,q^2;q^2\right)_\infty}{\left(\frac{b}{\lambda^2},\frac{\lambda^2q^2}{b},q,q,\frac{b}{q},\frac{q^3}{b},\lambda^2,\frac{q^2}{\lambda^2};q^2\right)_\infty}.$$

Proof. It can be seen from (7.5) and (7.1) that

$$\begin{aligned} g(b,\lambda,q) &= \sum_{n=1}^{\infty}\left(\frac{\lambda q^n}{1-\lambda^2q^{2n}} - \frac{\frac{\lambda}{b}q^n}{1-\frac{\lambda^2}{b^2}q^{2n}}\right) \\ &\quad + \sum_{n=1}^{\infty}\left(\frac{\lambda^2q^{2n}}{1-\lambda^2q^{2n}} + \frac{\frac{1}{q}q^{2n}}{1-\frac{1}{q}q^{2n}} - \frac{\frac{b}{q}q^{2n}}{1-\frac{b}{q}q^{2n}} - \frac{\frac{\lambda^2}{b}q^{2n}}{1-\frac{\lambda^2}{b}q^{2n}}\right) \\ &= f\left(\frac{\lambda}{b},\lambda,-1,q\right) + f\left(\frac{b}{q},\lambda^2,b,q^2\right), \end{aligned}$$

after setting

$$\frac{\lambda q^n}{1-\lambda q^n} = \frac{\lambda q^n}{1-\lambda^2q^{2n}} + \frac{\lambda^2q^{2n}}{1-\lambda^2q^{2n}}.$$

By similar reasoning,

$$g\left(\frac{1}{b},\frac{1}{\lambda},q\right)=f\left(\frac{b}{\lambda},\frac{1}{\lambda},-1,q\right)+f\left(\frac{q}{b},\frac{1}{\lambda^2},\frac{1}{b},q^2\right)-\frac{q}{1-q}+\frac{q/b}{1-q/b}.$$

The result now follows upon employing (7.3) and (7.4), after setting

$$\frac{(b/q-\lambda^2)(1-1/b)(1-\lambda^2/q)}{(1-b/q)(1-\lambda^2)(1-1/q)(1-\lambda^2/b)}+\frac{(\lambda/b-\lambda)(1+\lambda^2/b)}{(1-\lambda^2/b^2)(1-\lambda^2)}$$
$$-\frac{q}{1-q}+\frac{q/b}{1-q/b}=\frac{(1-b)\lambda\left(b^2-\lambda^3\right)}{(1-\lambda)\left(b-\lambda^2\right)\left(b^2-\lambda^2\right)}$$

□

COROLLARY 7.3. If $|q|<1$, then

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+3})(q^4;q^4)_n(-q)^n}{(1-q^{2n+1})(1-q^{2n+2})(q^2;q^4)_{n+1}}=\psi^4(q^2). \tag{7.6}$$

PROOF. Replace $q$ with $q^2$ in (7.2) and set $\lambda=1/q$. The left side of (7.6) follows after re-indexing and dividing through by $-q$. The right of (7.2) with the same substitutions and then divided by $-q$ simplifies to

$$\frac{1}{q}\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}.$$

The result now follows from [**4**, Example (iii), p. 139]:

$$q\psi^4(q^2)=\sum_{k=0}^{\infty}\frac{(2k+1)q^{2k+1}}{1-q^{4k+2}}.$$

□

## 8. Concluding remarks

One side of some of the identities has a combinatorial interpretation in terms of representations by quadratic forms. For example, the coefficient of $q^{2n}$ on the right side of (7.6) is equal to the number of representations of $n$ as a sum of four triangular numbers. A natural question is to ask if the left side of (7.6) also has an interesting combinatorial interpretation so that (7.6) encodes a combinatorial identity. Similar questions can be asked about some of the other identities in the paper. We leave these questions to the ingenuity of the reader.

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