McKibben Webster – Chapter 4 Partial Solutions and Hints

4.1.1 $\begin{cases} \dfrac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{cases}$

Let $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then, this system can be written more succinctly as

$$\begin{cases} \mathbf{u}'(t) = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\mathbf{u}(t) \\ \mathbf{u}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{cases}$$

4.3.1 A scalar multiple of a square matrix is a square matrix with the same dimensions, as are powers of such matrices. Also, sums of matrices are computed entrywise. So what?

4.3.2 $\begin{bmatrix} e^{a_{11}t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{a_{NN}t} \end{bmatrix}$. Tell why.

4.3.3 Use Definition 4.3.1 directly. Compute $\left(P^{-1}DP\right)^k$. How does this simplify? So what?

4.3.4 Differentiate using the chain rule and then use l'Hopital's rule to compute the limit as $n \to \infty$.

4.3.5 Mimic the approach used in Example 4.3.1 carefully.

4.3.6 (i) Use the usual inverse formula for a 2x2 matrix.

(ii) Argue inductively.

4.4.1 (i) $\begin{bmatrix} \cos(\alpha t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\alpha t) \end{bmatrix}$

4.5.1 The matrices A and αA commute. How do you use this with the definition when simplifying $e^{(\alpha A)t}e^{At}$?

4.5.2 (i) Use Example 4.3.1 to compute e^{At}. Then, simplify the quantity at which you are calculating the limit, and then compute the limit componentwise.

(ii) The power series for e^{At} is convergent and so, you can apply the linearity properties of convergent series. So, you can bring the *A* inside the summation. So what?

4.5.3 The integral of a matrix-valued function is performed entrywise. All matrices involved in both parts are diagonal, which makes the computations easy. Tell how.

4.7.1 (i) Let $U = \begin{bmatrix} y \\ y' \end{bmatrix}$ (using the change of variable trick). Use Putzer's algorithm to get

$$U(t) = \begin{bmatrix} 2e^{-\frac{1}{2}t} + 2\beta\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) \\ e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} + 2\beta e^{\frac{1}{2}t} - \beta e^{-\frac{1}{2}t} \end{bmatrix}$$

The solution *y*(*t*) is the first row of this vector.

(ii) $\beta = 0$ only. Tell why.

4.7.2 (i) $\alpha < 1$

(ii) $\alpha > 1$

4.7.3 (i) $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$

(ii) $t = -\ln\left(\frac{12+2\beta}{12+3\beta}\right)$, assuming this value lies in [0,1]. Also, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(iii) $\beta \geq -6$

4.8.1 Show that

$$\left|u(t) - u_{\varepsilon\delta}(t)\right| \leq e^{a(T-t_0)}\left[\left|1 - e^{\varepsilon(T-t_0)}\right|\left|u_0\right| + \left|\delta\right| e^{\varepsilon(T-t_0)}\right]$$

and then conclude that the desired limit is zero.

4.8.2 (i) This follows immediately from (4.67)

(ii) Apply the power series definition to compute all exponential matrices involved. The result follows from how matrices are multiplied.

(iii) Compute these using Putzer's algorithm.

4.8.3 (ii) This enables you to perform the operations used in (4.67). So what?

4.8.4 Mimic the development of the result following Exercise 4.8.1, step by step. It is sufficient to assume that $AB_\varepsilon = B_\varepsilon A$, where $A = \begin{bmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{bmatrix}$ and $B_\varepsilon = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \overline{\varepsilon_1} & \overline{\varepsilon_2} \end{bmatrix}$.

4.9.1 For instance…

(ii) $\alpha, \beta > 0 \Rightarrow$ solution curves become unbounded

(iii) $\alpha = 0,\ \beta > 0 \Rightarrow$ solution curves become unbounded for all nonzero initial conditions

(iv) $\alpha = 0,\ \beta < 0 \Rightarrow$ limit exists, so the behavior exhibited by the solution curves is stable.

4.9.2 (i) $\left\| e^{At} \right\|_{M^2}^2 = \sum_{k=1}^{N} \left(e^{\lambda_k t} \right)^2 \to 0,$ as $t \to \infty$ since $\lambda_k < 0$ for all values of k

(ii) If there exists k_0 such that $\lambda_{k_0} > 0,$ then $\left\| e^{At} \right\|_{M^2}^2 \to \infty$ (Tell why.) If there exists k_0 such that $\lambda_{k_0} = 0,$ and $\lambda_k < 0,$ for all other $k,$ then $\left\| e^{At} \right\|_{M^2}^2 \geq 1$. So, $\lim_{t \to \infty} \left\| e^{At} \right\|_{M^2}^2 \neq 0$.

4.10.1 The presence of the nonzero right side is the new wrinkle!

4.10.2 Add $\mathbf{F}(t)$ to the right side of (HCP)