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*Homogenous Case*

1. Consider a particle of mass m lying on a horizontal surface. The particle is connected to one end of a spring, and the other end of the spring is attached to a rigid wall at the end of the horizontal surface. Furthermore, assume the surface is frictionless. (See diagram).



 This spring mass system satisfies the assumption of Hooke’s Law from Newtonian physics. The following is a statement of Hooke’s Law: Suppose that the mass is pulled to the right with a force of magnitude Fo, increasing the tension in the spring. The mass is extended a distance x (usually in meters) from its equilibrium position. Hooke’s Law states that the restoring force of the spring, F (in Newtons), is proportional to the distance the mass was extended from its equilibrium position. Particularly, F= -kx, where k is a fixed spring constant with units N/m, characteristic of the spring under consideration. As the displacement increases, the restoring force in the direction opposite of displacement increases in magnitude.

 Note that the accuracy with which Hooke’s Law describes the spring’s motion depends on the magnitude of the mass’s displacement. (It is accurate for small displacements.)

 Let x(t) be a function of time that gives the position of the mass at time t. x=0 when the particle is at its equilibrium position, x>0 when the particle is to the left of equilibrium, and x<0 when it is to the right of the equilibrium point.

 Newton’s Second Law of Motion states that the net-force on a particle equals the mass (in kilograms) of a particle times its acceleration. (F=ma). As the restoring force is the only force acting on the acting on the particle after t=0, and the acceleration of a particle at time t with position function x(t) is given by the second derivative of the position function, x”(t), Newton’s 2nd Law and Hooke’s Law together imply that mx”(t)=-kx(t). This is a homogenous second order linear ODE. In order to specify a unique solution to this ODE, an initial position and velocity must be prescribed. For simplicity, and adherence to actual physical conditions, consider the initial position and velocity at t=0, and let x(0)=xo, and x’(0)=x1. Assuming a solution of the form x(t)= eλt , substitution, analysis of the characteristic equation, and Euler’s formula yields x(t)=C1(cos(sqrt(k/m)t)+C2(sin(sqrt(k/m)t). Applying the initial conditions yields x(t)=xocos(sqrt(k/m)t), which describes oscillatory behavior over time, as expected from physical observation.

Now consider a spring hanging from a stationary ceiling, a mass hanging from its end. The only addition to the net forces from the horizontal model is the force of gravity. This yields the ODE mx”(t)=mg-kx(t). A less accurate though easier model omits the force of gravity, yielding mx”(t)= -kx(t) , the same as the case of a horizontal spring.

The systems considered above are the simplest describing oscillation of a particle attached to a fixed horizontal and vertical spring. Greater complexity arises (in both accurate modeling and solution) when multiple masses are connected via multiple springs. Solutions of models of this nature are expedited by expressing them in the form of a Cauchy Problem.

A physical system of two masses and two springs is illustrated below.

The uppermost spring is fixed to a stationary ceiling, and has spring constant k1. At the end of this spring is a particle (call it p­1) of mass m1. From this particle hangs a second spring with spring constant k2. At the end of this spring hangs a particle of mass m2 (call it p2).

The goal is to derive an ODE that describes the motion of the particles after the second mass is pulled downward, then released. As in the case of a single particle, the initial displacement of both particles from equilibrium is assumed small, so that the motion is described accurately by Hooke’s Law.

Consider first p1. The second mass is pulled down, and then released. As the uppermost spring obeys Hooke’s Law, the upward restoring force on the particle by the top spring is given by –kx1 where x1=x1(t) is the displacement of the top particle from equilibrium. The restoring force of the lower spring on the top particle is given by –k2 (x1-x2), where x2 =x2 (t) is the displacement of the lower particle from equilibrium, and x2-x1 is the elongation of the second spring. Therefore, the net force on p1 is F= -kx1 –k2 (x1–x2). ­The net force on the lower particle is simply the restoring force of the lower spring, which is ­­-k2 (x2 –x1). Note that the restoring force of the lower spring on the top particle is opposite its force on the lower particle.

Therefore, applying Newton’s Second Law yields the following system of ODEs in two variables.

m1 x1”(t)= -k1x1(t)-k2 (x1(t)-x2(t))

m2 x2”(t)= -k2(x2(t)-x1(t)), t >0

Now consider a two-mass, three spring (2M-3S) system. (See illustration below).



The system of ODEs, whose solutions describe the motion of the two masses, is given by:

 m1 x1”(t)= -k1x1 (t)+k2(x2(t)-x1(t))

m2 x2”(t)=-k3x2(t)-k2 (x2 (t)-x1(t)), t > 0

Explanation: Consider particle of mass m­1 (call it p1). Let p1 be displaced from equilibrium a distance of x1, where x1 is small. Once more, Hooke’s Law dictates that the spring with spring constant k1 exerts a restoring force F1 = -k1x1 on p1. The middle spring exerts a restoring force F2 = -k2 (x1 – x2) on p1, where the elongation of this spring is the difference in the displacement of p1 and the second particle p2 from equilibrium. The net force acting on p1 is given by F= -kx1 +k2 (x2 –x1), and by Newton’s Second Law, the first ODE is obtained. The derivation of the ODE describing the motion of particle two is essentially the same, except the restoring force exerted by the middle spring on p2 is negative that which it exerts on p1.

1. Derivation of Homogenous Cauchy Problem for HSM (2M-2S) and (2M-3S)

We know that the system of two masses and two springs has the following system of ODEs to describe the motion of the two masses (or particles).

m1 x1”(t)= -k1x1(t)-k2 (x1(t)-x2(t))

m2 x2”(t)= -k2(x2(t)-x1(t)), t >0

Using algebra to isolate the second derivative, and separate the position functions of the two particles in each equation yields the following equivalent form of the above system.

x1”(t)= -((k1 + k2) /m1)x1(t) + (k2/m1)x2(t)

x2”(t)= (k2/m2)x1(t) -(k2 /m2)x2(t) , t > 0

Prescribe the following initial conditions: x1 (0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

For each function, the value of the function at zero provides the initial position of each mass at the moment the spring is released, after being extended. The derivative of each function gives the initial velocity of each particle at this same time.

Consider HCP: U’(t)=AU(t)

 U(0)=Uo , t > 0

Where U: [0, ∞) R4 , and A is a 4x4 coefficient matrix.

Let U(t)= Then U’(t)= and the original system of equations can be rewritten as

U’(t)= U(t)

U(0)=Uo = , t > 0

This system has a solution of the form U(t)= eAt Uo

Derivation of Homogenous Cauchy Problem for HSM (2M-3S)

For a system of two masses and three springs on a frictionless surface, the following system of ODEs has solutions describing the motion of the two masses.

m1 x1”(t)= -k1x1 (t)+k2(x2(t)-x1(t))

m2 x2”(t)=-k3x2(t)-k2 (x2 (t)-x1(t)), t > 0

Algebraic manipulation yields the following equivalent form of the system.

x1”(t)= -((k1 +k2 )/m1)x1(t)+(k2/m1)x2(t)

x2 ”(t)=(k2/m2)x1(t)-((k2+k3)/(m2))x2(t) , t > 0

Prescribe the following initial conditions: x1(0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

Consider once more HCP: U’(t)=AU(t)

 U(0)=Uo , t>0

Where U: [0, ∞) R4 and A is a 4x4 coefficient matrix

Let U(t)= Then U’(t)=

and

 U’(t)= U(t)

U(0)=Uo = , t >0

This system has a solution of the form U(t)= eAt Uo

1. Solutions of Homogenous CP Equations

Each of the formulated HCPs (for (2M-2S-HSM) and (2M-3S-HSM)) has a unique solution of the form U(t)=eAtUo , by Theorem 4.6. The solutions to each of the original IVPs are x1(t) and x2(t), which correspond to the first and third components of U(t) in each case.

Now consider the case of (nM-nS-HSM), a vertically hanging spring mass system with n springs and n masses. Extrapolating from the case of (2M-2S-HSM) yields the following system of ODEs

m1x1”(t)= -k1x1-k2(x1-x2)

m2x2”(t)= k2(x1-x2)-k3(x2-x3)

m3x3”(t)=k3(x2-x3) – k4(x3-x4)

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mnxn”(t)=kn(xn-1­ – xn )

Just as the system (2M-2S-HSM) could be reformulated as a HCP with a solution in R4, this system can be reformulated as a HCP with a solution in R2n. The solutions to the original system of equations correspond to the odd-number entries of the solution vector for HCP.

Consider the case of (nM-(n+1)S-HSM), a horizontal spring mass system with n+1 springs and n masses. Extrapolating from the case of (2M-3S-HSM) yields the following system of ODEs

m1x1”(t) = -k1x1+k2(x2-x1)

m2x2”(t) = -k2(x2-x1) + k3(x3-x2)

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mnxn”(t) = -kn+1(xn)-k­n(xn-xn-1)

As in the case of (nM-nS-HSM), this system can be reformulated as a HCP with solution in R2n , from which the solution to the orginal system may be extracted.

1. Continuous Dependence of (2M-2S-HSM) and (2M-3S-HSM)

Each of these HSM can be written in the form HCP. Continuous dependence of a solution to HCP on its initial conditions is investigated as follows.

First, consider (2M-2S-HSM). The HCP form of this spring model is given by

U’(t)= U(t)

U(0)= , t >0

Suppose the measurements of the initial conditions (the initial positions and velocities of the two masses) are subject to error.

Let Uε(0)= be the actual initial state vector, where | is a small positive number, for each.

Let Uε(t) be the solution of the HCP for (2M-2S) with Uε(0) replacing U(0).

Consider the solutions U(t) and Uε(t) for time interval 0 ≤ t ≤ T, and let | be the maximum | . Then U(t) is continuously dependent on the initial data over this time interval if

max || U(t)- Uε(t) || 0 as | 0 for 0 ≤ t ≤ T

Having proven continuous dependence for a general solution to HCP, and shown that (2M-2S-HSM) can be formulated as HCP, the solution to (2M-2S-HSM) must depend continuously on the initial data. For the same reason, (2M-3S-HSM) depends continuously on the initial data.

1. Influence of values of masses and spring constants for fixed initial conditions.

The following experiments were conducted using the MATLAB Linear ACP Solver GUI.

First, consider (2M-2S-HSM) with the following fixed initial conditions: x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0

1. Let k1 =45 N/m , k2=42 N/m, m1=5 kg, and m2= 4 kg. The solution of this system is graphed below from t=0 seconds until t=10 seconds.
2. Let k1=70 N/m, k2=68 N/m, m1= 5 kg, m2=4 kg. (Change only the spring constants.)

From comparing the graphs of the solutions with each of these parameters, I infer that increasing the value of the spring constants increases the frequency of oscillations of the solutions for fixed time and initial conditions. The graphs are given below.

 

The left graph gives solutions to (2M-2S-HSM) with spring constant values given in (i) The right graph shows solutions to the same model with the spring constant values given in (ii).

Now consider the effect of changing the mass values (with the same initial conditions).

1. Let k1=45 N/m, k2=42 N/m, m1=5 kg and m2=4 kg.
2. Let the spring constants have the same values. Let m1 = 10 kg and m2=8 kg

It appears that increasing the values of the masses decreases the frequency of oscillations of the solutions for fixed time and initial conditions. (See the graphs below)

 

The left graph gives the solutions to (2M-2S-HSM) with the mass values given in (i), and the right graph gives solutions to the same system of ODEs with the mass values given in (ii).

1. HSMD (HSM with dampening term proportional to the velocity of the masses.)

6.3

(2S-2M-HSMD) is given by the following alteration of (2S-2M-HSM)

m1 x1”(t)= -k1x1(t)-k2 (x1(t)-x2(t))-r1x1’(t)

m2 x2”(t)= -k2(x2(t)-x1(t))-r2x2’(t), t >0

where r1 and r2 are the proportionality constants for the resistance forces on the two masses.

Rearrangement of the terms yields the following equivalent system:

x1”(t)= -((k1 + k2)/m1)x1(t) + (k2/m1)x2(t)-(r1/m1)x1’(t)

x2”(t)= (k2/m2)x1(t) -(k2 /m2)x2(t)-(r2/m2)x2’(t) , t > 0

As in the case of reformulating (2S-2M-HSM),

 Let U(t)= so U’(t)=

Assume the same ICs: x1 (0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

This yields the following HCP for the dampened spring mass model:

U’(t)= U(t)

U(0)= , t > 0

As (2M-2S-HSMD) can be reformulated as HCP, it has a unique solution.

Does (2M-3S-HSMD) have a solution? To prove it does, reformulate the model as a HCP.

First consider the dampened model.

m1 x1”(t)= -k1x1 (t)+k2(x2(t)-x1(t))-r1x1’(t)

m2 x2”(t)=-k3x2(t)-k2 (x2 (t)-x1(t))-r2x2’(t) , t > 0

Rearrangement of the terms yields:

x1”(t)= -((k1 +k2 )/m1)x1(t)+(k2/m1)x2(t)- (r1/m1)x1’(t)

x2 ”(t)=(k2/m2)x1(t)-((k2+k3)/(m2))x2(t)-(r2/m2)x2’(t) , t > 0

Let U(t)= so U’(t)=

Assume the following ICs: x1 (0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

This yields the following HCP for the dampened spring mass model:

U’(t)= U(t)

U(0)= , t > 0

Reformulation of the model as HCP implies the existence of a solution to (2S-3M-HSMD)

As in the case of (nM-nS-HSM) and (nM-(n+1)S-HSM), each of the models (nM-nS-HSMD) and (nM-(n+1)S-HSMD) have solutions, as they too may be formulated as HCP, each with a solution vector in R2n . Here is the line of reasoning followed to reach this conclusion, without actual formulation of these systems as HCP. We know that (nM-nS-HSM) and (nM-(n+1)S-HSM) can be reformulated as HCP with solution vectors in R2n . When dampening terms were added to (2M-2S-HSM) and (2M-3S-HSM), the resulting models could also be formulated as HCP, with solution vectors also in R4. The dampening term can be incorporated into the coefficient matrix, as the solution vector conveniently includes the derivatives of the state functions for each mass.Therefore, (nM-nS-HSMD) and (nM-(n+1)S-HSMD) should be able to be reformulated as HCP with solution vectors in R2n . From this solution vector, the solutions to the original system of ODEs with dampening terms can be extracted. They are the odd-numbered components of the solution vector U(t) .

6.4

Given that 2M-2S-HSMD and 2M-3S-HSMD can be formulated as HCP, and the solution of a general equation in the form HCP depends continuously on its initial conditions, we conclude that each of these models depends continuously on its initial conditions. Rather than formulate the analytical process by which continuous dependence is proven, as in the case of HSM, in this case I will provide graphical evidence of continuous dependence on initial conditions.

The following continuous dependence experiments were conducted using the Linear ACP Solver GUI. First, consider (2M-2S-HSMD) with the following fixed spring constants and masses:

1. Let k1 =45 N/m , k2=42 N/m, m1=5 kg, m2= 4 kg, r­1=.5 and r2=.5. Continuous dependence is considered for 0≤ t ≤10. The initial conditions are originally x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0 m/s. The perturbed initial conditions are x1(0)=.06 m, x1’(0)=0.05 m/s, x2(0)=.10 m, x2’(0)=0.05 m/s. While the norm of the difference between the original and perturbed initial condition vectors was 0.076, the norm of the difference between the original and perturbed solution vectors was 0.18, suggesting that the solutions do not depend continuously on the initial data.
2. Let k1 =5 N/m , k2=4 N/m, m1=5 kg, m2= 4 kg, r­1=.5 and r2=.5. The point of this experiment is to consider the effect of smaller valued spring constants on continuous dependence. Continuous dependence is considered for 0≤ t ≤10. The initial conditions are originally x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0 m/s. The perturbed initial conditions are x1(0)=.06 m, x1’(0)=0.05 m/s, x2(0)=.10 m, x2’(0)=0.05 m/s.

While the norm of the difference between the original and perturbed initial condition vectors was 0.076, the norm of the difference between the original and perturbed solution vectors was .23, once more suggesting that the solutions do not depend continuously on the initial data.

It seems that regardless of the changes made to the values of the spring constants, the solutions still do not depend continuously on the initial data.

Now consider (2M-3S-HSMD)

1. Let k1 =45 N/m , k2=42 N/m, m1=5 kg, m2= 4 kg, r­1=.5 and r2=.5. Continuous dependence is considered for 0≤ t ≤10. The initial conditions are originally x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0 m/s. The perturbed initial conditions are x1(0)=.06 m, x1’(0)=0.05 m/s, x2(0)=.10 m, x2’(0)=0.05 m/s. Once more, the norm of the difference between the original and perturbed initial condition vectors was .29, and the norm of the difference between the original and perturbed solution vectors was over 10,000! (This clearly suggests an absence of continuous dependence on the initial data).

6.5 Conjecture how the masses and spring constants affect the motion of the masses for fixed initial conditions.

Consider (2M-2S-HSMD) with the following initial conditions: x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0

1. Let k1 =45 N/m , k2=42 N/m, m1=5 kg, m2= 4 kg, r1=0.5, r2=0.5. And observe the behavior of the solutions for the first ten seconds
2. Let k1=70 N/m, k2=68 N/m, m1= 5 kg, m2=4 kg, r1=0.5, r2=0.5 (Change only the spring constants.) Once again, observe the behavior of the solutions for the first ten seconds.

As in the case of (2M-2S-HSM), increasing the value of the spring constants increases the frequency of the oscillations of the solutions.

Consider (2M-3S-HSMD) with the following initial conditions: x1(0)=.04 m, x1’(0)=0 m/s, x2(0)=.08 m, x2’(0)=0

1. Let k1 =45 N/m , k2=42 N/m, k3=40 N/m m1=5 kg, m2= 4 kg, r1=0.5, r2=0.5. And observe the behavior of the solutions for the first ten seconds
2. Let k1=70 N/m, k2=68 N/m, k3=60, m1= 5 kg, m2=4 kg, r1=0.5, r2=0.5 (Change only the spring constants.) Once again, observe the behavior of the solutions for the first ten seconds.

Once more, increasing the value of the spring constants increases the frequency of the oscillations of the solutions.

To explore the dependence of (2M-2S-HSMD) on the values of the masses, consider the exact same initial conditions as above.

1. Let k1 =45 N/m , k2=42 N/m, m1=5 kg, m2= 4 kg, r1=0.5, r2=0.5. And observe the behavior of the solutions for the first ten seconds
2. Let k1=45 N/m, k2=42 N/m, m1= 10 kg, m2=8 kg, r1=0.5, r2=0.5 (Change only the values of the masses.) Once again, observe the behavior of the solutions for the first ten seconds.

As the values of the masses increased, the frequency of the oscillations of the solutions decreased.

To explore how (2M-3S-HSMD) is affected by changes in the values of the masses, I used the same values as used above, with k3=40 in both cases. As the values of the masses increased, the frequency of the oscillations of the solutions decreased.

*Non-Homogenous Case*

1. Consider (2M-2S-HSM) with forcing term mg in each ODE (g= -9.8 m/s). This is the natural forcing term for a vertically hanging spring, as the force of gravity acts upon each mass as it oscillates.

m1 x1”(t)= -k1x1(t)-k2 (x1(t)-x2(t))+m1g

m2 x2”(t)= -k2(x2(t)-x1(t))+m2g , t >0

(2M-2S-HSMD) with forcing term mg:

m1 x1”(t)= -k1x1(t)-k2 (x1(t)-x2(t))-r1x1’(t)+ m1g

m2 x2”(t)= -k2(x2(t)-x1(t))-r2x2’(t)+m2g , t >0

(2M-3S-HSM) with forcing term Fdcos(ωt) (where ω and Fd are real numbers) acting on each mass. Rather than considering the left and right most springs as being attached to a rigid wall, they may each be attached to a piston that oscillates back and forth. The force of the piston on each mass is expressed by the forcing term, which gives the period of the piston’s oscillation as 2π/ω and its amplitude as Fd.

m1 x1”(t)= -k1x1 (t)+k2(x2(t)-x1(t))+Fdcos(ωt)

m2 x2”(t)=-k3x2(t)-k2 (x2 (t)-x1(t))+ Fdcos(ωt) , t > 0

(2M-3S-HSMD) with forcing term Fdcos(ωt) acting on each mass.

m1 x1”(t)= -k1x1 (t)+k2(x2(t)-x1(t))-r1x1’(t)+ Fdcos(ωt)

m2 x2”(t)=-k3x2(t)-k2 (x2 (t)-x1(t))-r2x2’(t)+ Fdcos(ωt) , t > 0

2. Deriving non-HCP for each of the above models.

(2M-2S-HSM)

Simply add the forcing terms as a separate vector in R4 to (2M-2S-HSM) without the forcing term.

Consider non-HCP: U’(t)=AU(t) +F(t)

 U(0)=Uo , t > 0

 Where U: [0, ∞) R4 , F: [0, ∞) R4 and A is a 4x4 coefficient matrix.

Assume the following ICs: x1 (0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

Let U(t)= Then U’(t)= and the original system of equations can be rewritten as

U’(t)= U(t) +

U(0)=Uo = , t > 0

(2M-2S-HSMD) The simple amendment made above can be made once more to (2M-2S-HSMD) without the forcing term.

Let U(t)= so U’(t)=

This yields the following HCP for the dampened spring mass model:

U’(t)= U(t)+

U(0)= , t > 0

(2M-3S-HSM)

U’(t)= U(t)+

U(0)=Uo = , t >0

(2M-3S-HSMD)

U’(t)= U(t) +

U(0)= , t > 0

3. In each of the above non-HCPs, the forcing term is either or

We know that a vector-valued function in Rn is continuous if and only if each of its components is itself a continuous function in R. Clearly, the first forcing term, whose components are constants, is a continuous function in R4 . As the coefficient matrices for (2M-2S-HSM) and (2M-2S-HSMD) are in M4(R) and Uo is in R4 , by Theorem 5.3.1, both of these models, reformulated as non-HCP, have unique classical solutions. The second forcing term has continuous components as well. Clearly 0 is a continuous function in R, and as cos(x) is continuous, it follows that is continuous. Therefore, the second forcing term is also continuous. As (2M-3S-HSM) and (2S-3S-HSMD) satisfy the necessary hypotheses of Theorem 5.3.1, the non-HCPs given for these two models have unique classical solutions as well.

The solution to the general non-HCP satisfying the hypotheses of Theorem 5.3.1 is given by the variation of parameters formula.

U(t)=eA(t)Uo + for all t > 0 .

Regarding the existence a solution to the non-homogenous (nM-nS-HSMD) and (nM-(n+1)S-HSMD). These models can be formulated by amending (nM-nS-HSM) and (nM-(n+1)S-HSM) to account for both dampening terms and forcing terms.

Consider first (nM-nS-HSMD) with the necessary amendments.

m1x1”(t)= -k1x1-k2(x1-x2)-r1x1’(t)+m1g

m2x2”(t)= k2(x1-x2)-k3(x2-x3)-r2x2’(t)+m2g

m3x3”(t)=k3(x2-x3) – k4(x3-x4)-r­­3x3’(t)+m3g

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mnxn”(t)=kn(xn-1­ – xn )-rnxn’(t)+mng

As in the case of non-homogenous 2M-2S-HSMD, this system of n ODEs can be formulated as non-HCP, where all but the forcing term mg in each equation is incorporable into the coefficient matrix when the solution is assumed to be a vector in R2n of the form U(t)=. The forcing term is given by F(t)=

Therefore, nM-nS-HSMD can be formulated as non-HCP. This IVP satisfies the hypotheses of Theorem 5.3.1, and therefore a solution exists for nM-nS-HSMD in Rn.

The same line of reasoning leads to the conclusion that nM-(n+1)S-HSMD has a solution in Rn .

4. Investigate the dependence of each model on the initial conditions and the forcing term. Having formed a general result for the continuous dependence of any classical solution of non-CP on the initial data, we know *a priori* that each of the above models indeed depends continuously on the initial data, as they each can be written as a non-HCP that has a classical solution.

5. Are there forcing terms that can cause the displacement of the masses to grow in time? The Linear ACP Solver is used to investigate this question.

Consider non-homogenous (2M-2S-HSM). With initial conditions x1 (0)=.02m, x1’(0)=0m/s , x2(0)=.04m, x2’(0)=0 m/s

Let k1 =45 N/m, k2=42 N/m, k3=40 N/m m1=5 kg, m2= 4 kg. Consider time from t=0 until t=20 seconds. If the forcing term of non-HCP is (meaning the forcing term of each ODE in the original model is et ­), then as time passes, the displacement of the masses increases exponentially. (See graph below).

Using these same initial conditions and values of parameters, a forcing term of the form also causes the displacement of the masses to increase uniformly, though the rate of displacement is less than when the forcing terms were exponential. Presumably then, a forcing term of the form where n is a natural number greater than or equal to two, also causes the displacement of the masses to grow with time.

 

Solutions to 2M-2S-HSM with forcing terms f(t)=e^t and f(t)=t2

*Semi-Linear Case*

1. Consider once more 2M-2S-HSM, 2M-3S-HSM, 2M-2S-HSMD and 2M-3S-HSMD, with the term –kx(t) replaced by –kx(t)+μx3(t) in each model.

(2M-2S-HSM)

m1 x1”(t)= -k1x1(t)+ μx1(t)3-k2 (x1-x2)+μ(x1-x­2)3+m1g

m2 x2”(t)= -k2(x2-x1)+μ(x2-x1)3+m2g , t > 0

(2M-2S-HSMD)

m1 x1”(t)= -k1x1(t)+ μx1(t)3-k2 (x1-x2)+μ(x1-x­2)3+m1g-r1x1’(t)

m2 x2”(t)= -k2(x2-x1)+μ(x2-x1)3+m2g-r2x2’(t) , t > 0

(2M-3S-HSM)

m1x1”(t)=-k1x1+ μx13+k2(x2-x1)- μ(x2-x1)3+ Fdcos(ωt)

m2x2”(t)=-k3x2+ μx23 –k2 (x2-x1)+(x2-x1)3 + Fdcos(ωt)

(2M-3S-HSMD)

m1x1”(t)=-k1x1+ μx13+k2(x2-x1)- μ(x2-x1)3-r1x1’(t)+ Fdcos(ωt)

m2x2”(t)=-k3x2+ μx23 –k2 (x2-x1)+ μ(x2-x1)3 –r2x2’(t)+ Fdcos(ωt)

1. Derive Semi-Linear CP for each model.

(2M-2S-HSM)

Consider Semi-Linear CP: U’(t)=AU(t) +F(t, U(t))

 U(0)=Uo , t > 0

 Where U: [0, ∞) R4 , F: [0, ∞) x R4 R4 and A is a 4x4 coefficient matrix.

Assume the following ICs: x1 (0)=x10 , x1’(0)=x11 , x2(0)=x20 , x2’(0)=x21

Let U(t)= Then U’(t)= and the original system of equations can be rewritten as

U’(t)= U(t) +

U(0)=Uo = , t > 0

(2M-2S-HMSD)

U’(t)= U(t) +

U(0)=Uo = , t > 0

(2M-3S-HSM)

U’(t)= U(t)+

U(0)=Uo = , t >0

(2M-3S-HSMD)

U’(t)= U(t) +

U(0)=Uo = , t >0

1. Conditions for which the semi-linear CP possesses a solution.

In order for each semi-CP to posess a **unique mild** solution, the following conditions must hold. The coefficient matrix must be a 4x4 square matrix with real value entries, and Uo must be an R4 valued vector. Furthermore, the forcing function F:[0,T) x R4  R4 must be continuous in t and uniformly Lipschitz in the spatial variables.

1. The solution in *Non-Homogenous Case* question 5 is an example of a solution causing the displacement of the masses to grow with time. This is caused by forcing terms of an exponential or polynomial nature (of sufficiently high degree). The terms are not nonlinear however, involving neither of the unknown functions. The solutions of the models in question 1 have non-linear forcing terms, but do not increase unbounded in time. The Lipschitz condition on the forcing term in each semi-linear CP limits the rate of growth of the solution function with time to ensure the existence of a unique mild solution. Engineers that want to model spring mass systems with non-linear forcing terms that cause the displacement of the masses to grow in time at sufficiently high rates cannot formulate such models as semi-linear CP.

Work Cited

Davis, Doug. "Periodic Motion." Periodic Motion. Eastern Illinois University, 2002. Web. 1 May 2015. <http://www.ux1.eiu.edu/~cfadd/1150/15Period/Vert.html>.

Fay, Temple H., and Sarah Duncan Graham. "Coupled Spring Equations." International Journal of Mathematical Education in Science and Technology 34.1 (2003): 65-79. Web.

Logan, J. David. A First Course in Differential Equations. New York: Springer, 2006. Print.

McKibben, Mark A., and Micah D. Webster. Differential Equations with MATLAB: Exploration, Applications, and Theory. Boca Raton: CRC, 2015. Print.

Morin, David. Normal Modes. Milton Keynes: Open UP, 1982. Web. 1 May 2015. <http://www.people.fas.harvard.edu/~djmorin/waves/normalmodes.pdf>.