IRRATIONAL LINEAR FORMS IN PRIME VARIABLES

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Abstract. We apply a recent refinement of the Hardy-Littlewood method to obtain an asymptotic lower bound for the number of solutions of a linear diophantine inequality in 3 prime variables. Using the same ideas, we are able to show that a linear form in 2 primes closely approximates almost all real numbers lying in a suitably discrete set.

1. Introduction

Suppose that F is a form with real coefficients, not all in rational ratio. When F is diagonal and indefinite, a version of the Hardy-Littlewood method developed by Davenport and Heilbronn [6] allows one to demonstrate that the values of F at integral points are dense in the real line, provided that the number of variables is sufficiently large in terms of the degree. A quite different method of Margulis [9] in fact establishes such a conclusion for arbitrary indefinite quadratic forms in three variables, and work of Schmidt [12] allows one to consider non-diagonal forms of odd degree. The Davenport-Heilbronn method seeks to exploit the irrationality of F to show that the relevant product of exponential sums has its only substantial peak near the origin, and one typically achieves this by restricting the main parameter to a sequence determined by the denominators occurring in the continued fraction expansion of one of the coefficient ratios. While this approach is often sufficient to demonstrate that there are infinitely many solutions $\mathbf{x}$ to an inequality of the shape $|F(\mathbf{x}) - \mu| < \eta$, one drawback is that it does not yield the expected estimate for the number of solutions lying in a box unless the box size happens to coincide with an element of the sequence mentioned above. In particular, the method does not give estimates that are valid for all sufficiently large boxes, as one normally desires.

When the form F is definite, its values will not be dense, but one may instead hope to establish that the gaps between its values tend to zero near infinity. The type of strategy described above, however, would yield at best the conclusion that there exist arbitrarily small gaps between values. To recover control of the limiting process, one often imposes a hypothesis that certain coefficient ratios are algebraic (or badly approximable). Then, without restricting the parameter governing the box size, one can show that abnormally large exponential sums would lead to rational approximations that are incompatible with such a hypothesis, except possibly

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on a set of very small measure. This restriction to algebraic coefficients, while undesirable, has in fact been imposed in a number of papers dealing not only with the topic at hand [4, 7] but also with systems of diophantine inequalities [3, 5, 11].

A recent innovation of Bentkus and Götze [2] provides a remedy for both types of defects mentioned above. One of the main results of [2] deals with gaps between values of positive definite quadratic forms, and further development of these ideas by Freeman [8] yields asymptotic estimates for the number of solutions of inequalities involving indefinite forms. We seek here to apply this new method to provide similar conclusions concerning the values of irrational linear forms at prime arguments. As with the conventional version of Goldbach's problem, one can make interesting statements about both binary and ternary forms.

Our first result deals with indefinite forms in three primes. Let $N(X;\eta)$ denote the number of solutions of the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 - \mu| < \eta \tag{1}$$

in primes p_1, p_2, p_3 not exceeding X. The following theorem provides the expected lower bound for $N(X;\eta)$.

Theorem 1. *Suppose that λ_1, λ_2, and λ_3 are non-zero real numbers, not all of the same sign, and not all in rational ratio. Then given any $\eta > 0$ and $\mu \in \mathbb{R}$, there is a number $X_0 = X_0(\boldsymbol{\lambda}, \mu, \eta)$ such that for $X \geq X_0$ one has*

$$N(X;\eta) \gg \eta X^2 (\log X)^{-3},$$

where the implicit constant depends at most on $\boldsymbol{\lambda}$.

We mention that Vaughan [13] has already shown that there are infinitely many solutions to (1), and in fact it is a simple matter to obtain such a result with η replaced by a negative power of $\log X$, where $X = \max(p_1, p_2, p_3)$. Vaughan's paper in fact demonstrates using zero-density estimates and a zero-free region for the Riemann zeta function that η can be replaced by the function $X^{-1/10}(\log X)^{20}$. Results of this type are not obtainable by the Bentkus-Götze-Freeman method, and it has been pointed out by A. Baker (see [14], exercise 11.4) that such improvements are in fact impossible in the situation of Theorem 1. That is, for any explicit function $\eta(X)$ tending to 0, there are choices of the λ_i, and arbitrarily large X, for which the inequality (1) becomes insoluble when η is replaced by $\eta(X)$. Thus one is forced to choose between a result that quantifies the rate of decay and one that captures the asymptotic density of solutions.

We now formulate a statement concerning positive binary forms in two primes. Given a set of positive real numbers $\mathcal{Z}$, we say that $\mathcal{Z}$ is η-spaced if every two distinct elements of $\mathcal{Z}$ differ by at least η. Write $\mathcal{E}(X;\mathcal{Z};\eta)$ for the set of all $z \in \mathcal{Z} \cap [1, X]$ for which the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 - z| < \eta \tag{2}$$

has no solution in primes p_1, p_2, and let $E(X;\eta)$ denote the supremum of $|\mathcal{E}(X;\mathcal{Z};\eta)|$ over all η-spaced sets $\mathcal{Z}$. The following result may be viewed as an approximation to the conjecture that the gaps between the values of $\lambda_1 p_1 + \lambda_2 p_2$ tend to zero near infinity.

Theorem 2. *Suppose that λ_1 and λ_2 are positive real numbers with λ_1/λ_2 irrational, and fix $\varepsilon > 0$. Then there is a function $\Phi(X)$, depending at most on $\boldsymbol{\lambda}$ and ε, such that $\Phi(X) = o(X)$ as $X \to \infty$ and*

$$E(X;\eta) \leq \eta^{-2-\varepsilon}\,\Phi(X)$$

for every η with $0 < \eta < 1$.

In fact, if $R(z;\eta)$ denotes the number of solutions of (2), then our methods may be adapted to give the expected lower bound $R(z;\eta) \gg \eta z(\log z)^{-2}$ for almost all z in a given η-spaced set. The conclusion of Theorem 2 should be compared with the superior bound of $\eta^{-2}X^{2/3+\varepsilon}$ obtained by Brüdern, Cook, and Perelli [4] in the special case where λ_1/λ_2 is algebraic. In that case, one can potentially obtain non-trivial information when η is nearly as small as $X^{-1/3}$, since the cardinality of $\mathcal{Z} \cap [1, X]$ may then be nearly as large as $X^{4/3}$. In the more general situation of Theorem 2, however, η must be taken somewhat larger than $\Phi(X)/X$, and we have no information about the rate at which this function approaches zero.

We recall for comparison that Montgomery and Vaughan [10] have obtained a bound of the form $X^{1-\delta}$ for the exceptional set in the binary Goldbach problem, and that this bound improves to $X^{1/2+\varepsilon}$ under the Generalized Riemann Hypothesis. Furthermore, it is noted in [4] that the bound in the algebraic case of Theorem 2 can be replaced by something slightly better than $\eta^{-2}X^{1/2+\varepsilon}$ under GRH. However, it seems that GRH would have no consequences so far as the general version of Theorem 2 is concerned. The weakness of our bound therefore appears to result mainly from issues in diophantine approximation rather than from poor information about primes in arithmetic progressions.

In light of the prime number theorem, it is convenient to consider the weighted exponential sum

$$f(\alpha) = f(\alpha; X) = \sum_{p \leq X} e(\alpha p)\log p. \tag{3}$$

Our success depends primarily on being able to establish a non-trivial estimate for the product $f(\lambda_1\alpha)f(\lambda_2\alpha)$ when λ_1/λ_2 is irrational and α lies in the region away from the origin but bounded above by a carefully defined function tending (perhaps very slowly) to infinity. Section 2 is devoted to establishing such a result and also to providing suitable estimates for $f(\alpha)$ in mean value. The proofs of Theorems 1 and 2 then follow with little difficulty in Sections 3 and 4, respectively.

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2. Estimates for the exponential sum over primes

We start with a lemma expressing the general philosophy that large exponential sums yield good rational approximations to the coefficients.

Lemma 1. *Suppose that $|f(\alpha)| \ge X/A$, where $A \le \log X$. Then there exist coprime integers q and a satisfying*

$$q \ll A^2 \qquad \textit{and} \qquad |q\alpha - a| \ll A^2 X^{-1}.$$

Proof. We may clearly suppose that $\alpha \in [0,1)$, for then the result extends to all real α by periodicity after adding a suitable multiple of q to a. Define a set of major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le LX^{-1}\} \tag{4}$$

with $(q,a) = 1$ and $0 \le a \le q \le L$, where $L = (\log X)^B$ and B is a sufficiently large constant. By Theorem 3.1 of Vaughan [14], one has

$$f(\alpha) \ll (\log X)^4 (Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}) \tag{5}$$

whenever $|q\alpha - a| \le q^{-1}$ and $(q,a) = 1$. If $\alpha \notin \mathfrak{M}$, then we may use Dirichlet's Theorem to obtain coprime integers q and a satisfying

$$L < q \le XL^{-1} \qquad \text{and} \qquad |q\alpha - a| \le LX^{-1},$$

so on taking $B \ge 12$, we see from (5) that

$$f(\alpha) \ll X(\log X)^{-2}. \tag{6}$$

Now suppose instead that $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$. Then it follows easily from [14], Lemma 3.1, that

$$f(\alpha) = \frac{\mu(q)}{\phi(q)} v(\alpha - a/q) + O(X \exp(-C\sqrt{\log X})) \tag{7}$$

for some constant C, where

$$v(\beta) = \int_1^X e(\beta u)\, du. \tag{8}$$

The bound

$$v(\beta) \ll X(1 + |\beta| X)^{-1} \tag{9}$$

is immediate, and we also recall that

$$\phi(q) \ge q \prod_{p \le q} \left(1 - \frac{1}{p}\right) \gg \frac{q}{\log 2q},$$

by Mertens' theorem. Therefore one has

$$f(\alpha) \ll \frac{X \log 2q}{q + |q\alpha - a| X} \tag{10}$$

whenever $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, since the expression on the right of (10) then dominates the error term in (7). On combining (6) and (10), we find that for any $\alpha \in [0,1)$, there exist coprime integers q and a for which

$$f(\alpha) \ll \frac{X}{(\log X)^2} + \frac{X \log 2q}{q + |q\alpha - a| X}.$$

Thus if $|f(\alpha)| \geq X/A$, where $A \leq \log X$, we see that

$$A^{-1} \ll \frac{\log 2q}{q + |q\alpha - a|X}.$$

It follows immediately that $q \ll A^{1+\varepsilon}$ and hence also that $|q\alpha - a| \ll A^{1+\varepsilon}X^{-1}$ for any $\varepsilon > 0$. □

We note that a version of Lemma 1, with an additional power of $\log X$ in the two bounds, would follow directly from (5) without incorporating major arc information. Such a result, however, would not be adequate for our purposes, as the proof of the following lemma exploits the case where A is a constant, and the result that q is then also bounded by a constant is crucial to the argument.

Lemma 2. *Suppose that B and T are fixed positive numbers, and write $\mathcal{I}(X)$ for the interval $[LX^{-1}, T]$, where $L = (\log X)^B$. Then whenever λ_1/λ_2 is an irrational number, one has*

$$\lim_{X\to\infty} \sup_{|\alpha| \in \mathcal{I}(X)} \frac{|f(\lambda_1\alpha)f(\lambda_2\alpha)|}{X^2} = 0.$$

Proof. If the statement is false, then we can find $\varepsilon > 0$ and sequences of real numbers $\{X_n\}$ and $\{\alpha_n\}$, with $X_n \to \infty$ and $|\alpha_n| \in \mathcal{I}(X_n)$, such that

$$|f(\lambda_i\alpha_n; X_n)| \geq \varepsilon X_n \qquad (i = 1, 2)$$

for each n. Thus by applying Lemma 1 with $A = 1/\varepsilon$, we obtain a sequence of integral 4-tuples $(q_{1n}, a_{1n}, q_{2n}, a_{2n})$ with

$$(q_{in}, a_{in}) = 1, \quad q_{in} \ll 1, \quad \text{and} \quad |q_{in}\lambda_i\alpha_n - a_{in}| \ll X_n^{-1} \quad (i = 1, 2), \tag{11}$$

where the implicit constants of course depend on ε. It follows that there are only finitely many distinct choices, depending on ε, T, λ_1 and λ_2, for the 4-tuple $(q_{1n}, a_{1n}, q_{2n}, a_{2n})$, and hence there is a 4-tuple (q_1, a_1, q_2, a_2) that occurs for infinitely many n. Moreover, since each α_n lies in the compact interval $[-T, T]$, there is a subsequence $\mathcal{S}$ of these latter n along which α_n converges to a limit, say α_0. If $\alpha_0 = 0$, then for sufficiently large n we have $|\alpha_n| \leq (2q_1|\lambda_1|)^{-1}$ and hence (11) implies that $a_1 = 0$. But then $\alpha_n \ll X_n^{-1}$ for large n, which is impossible since $|\alpha_n| \in \mathcal{I}(X_n)$. Hence we must have $\alpha_0 \neq 0$, so on letting $n \to \infty$ through $\mathcal{S}$, we see from (11) that

$$q_i\lambda_i\alpha_0 = a_i \qquad (i = 1, 2),$$

and thus $\lambda_1/\lambda_2 = a_1q_2/(a_2q_1)$, contradicting the hypothesis that λ_1/λ_2 is irrational. □

Before proceeding to the proofs of our theorems, we record an optimal mean value estimate for $f(\alpha)$.

Lemma 3. *Whenever $s > 2$, one has*

$$\int_0^1 |f(\alpha)|^s \, d\alpha \ll X^{s-1}.$$

Proof. First of all, by orthogonality and the prime number theorem, one has

$$\int_0^1 |f(\alpha)|^2 \, d\alpha \ll X \log X. \tag{12}$$

We now consider a Hardy-Littlewood dissection as in the proof of Lemma 1. Define $\mathfrak{M}$ as in (4), and put $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. Write $s = 2 + \delta$, where $\delta > 0$, and set $B = 4/\delta + 8$. Then by (5) and the argument leading to (6), one has

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll X (\log X)^{-2/\delta},$$

which in combination with (12) gives

$$\int_{\mathfrak{m}} |f(\alpha)|^s \, d\alpha \ll \left(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \right)^{\delta} \int_0^1 |f(\alpha)|^2 \, d\alpha \ll X^{s-1} (\log X)^{-1}.$$

Moreover, by (10), one has

$$\int_{\mathfrak{M}} |f(\alpha)|^s \, d\alpha \ll X^s \sum_{q \le L} (\log 2q)^s q^{1-s} \int_0^\infty \frac{d\beta}{(1+\beta X)^s} \ll X^{s-1},$$

and the lemma follows. □

Of course, it is immediate from (12) that the mean value considered in Lemma 3 is bounded by $X^{s-1} \log X$, but such a result does not suffice in our situation because of the lack of quantitative information available in Lemma 2.

3. Indefinite ternary forms

Given the estimates of the previous section, the proof of Theorem 1 is essentially a routine exercise. Before starting the analysis, we make some simplifying reductions. First of all, by relabeling variables, we may clearly suppose that λ_1/λ_2 is irrational. If $\lambda_1/\lambda_2 > 0$, then both λ_1/λ_3 and λ_2/λ_3 must be negative, and at least one of these must be irrational. Thus by further relabeling, we may suppose that λ_1/λ_2 is irrational and negative. Additionally, we may replace each λ_i by $-\lambda_i$ and μ by $-\mu$ if necessary to ensure that $\lambda_3 > 0$, and we may suppose after possibly one more switch that $\lambda_1 > 0$ and $\lambda_2 < 0$. Finally, after dividing both sides of (1) by $\min(\lambda_1, |\lambda_2|, \lambda_3)$, we may assume that for each i one has $|\lambda_i| \ge 1$.

We suppose throughout that X is sufficiently large in terms of $\boldsymbol{\lambda}$, μ, and η. We adopt the notation $f_i(\alpha) = f(\lambda_i \alpha)$, where $f(\alpha)$ is as in (3), and we introduce the kernel

$$K(\alpha) = K(\alpha; \eta) = \left(\frac{\sin \pi \eta \alpha}{\pi \alpha} \right)^2. \tag{13}$$

A simple calculation (see for example [1], Lemma 14.1) reveals that

$$\widehat{K}(t) = \int_{-\infty}^{\infty} K(\alpha) e(\alpha t) \, d\alpha = \max(0, \eta - |t|), \tag{14}$$

and we note also the obvious bound

$$K(\alpha) \ll \min(\eta^2, |\alpha|^{-2}). \tag{15}$$

When $\mathfrak{B} \subseteq \mathbb{R}$, we define the integral

$$I_{\mathfrak{B}}(X;\eta) = \int_{\mathfrak{B}} f_1(\alpha) f_2(\alpha) f_3(\alpha) e(-\alpha\mu) K(\alpha)\, d\alpha.$$

Then in view of (14) we have

$$\begin{aligned} I_{\mathbb{R}}(X;\eta) &= \sum_{p_1,p_2,p_3 \le X} \widehat{K}(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 - \mu) \log p_1 \log p_2 \log p_3 \\ &\le \eta (\log X)^3 N(X;\eta). \end{aligned}$$

In order to prove Theorem 1, it therefore suffices to establish the estimate

$$I_{\mathbb{R}}(X;\eta) \gg \eta^2 X^2, \tag{16}$$

where the implicit constant depends at most on $\boldsymbol{\lambda}$.

Write $L = (\log X)^B$, and define the major arc to be the interval

$$\mathfrak{M} = \{\alpha : |\alpha| \le L X^{-1}\}. \tag{17}$$

We first obtain a lower bound for $I_{\mathfrak{M}}(X;\eta)$. When $\alpha \in \mathfrak{M}$, one has by (7) that

$$f(\alpha) = v(\alpha) + O(X \exp(-C\sqrt{\log X})),$$

where $v(\alpha)$ is as in (8). Let us write $v_i(\alpha) = v(\lambda_i \alpha)$. Since $\operatorname{meas}(\mathfrak{M}) \ll L X^{-1}$, it follows easily that

$$I_{\mathfrak{M}}(X;\eta) = \int_{\mathfrak{M}} v_1(\alpha) v_2(\alpha) v_3(\alpha) e(-\alpha\mu) K(\alpha)\, d\alpha + O(X^2 \exp(-C_1\sqrt{\log X})).$$

Furthermore, on using (8) and (9), we find that

$$I_{\mathfrak{M}}(X;\eta) = J(X;\eta) + O(X^2 (\log X)^{-2B}),$$

where

$$J(X;\eta) = \int_{-\infty}^{\infty} \int_{[1,X]^3} e((\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 - \mu)\alpha) K(\alpha)\, d\mathbf{u}\, d\alpha.$$

Now on interchanging the order of integration, we see that

$$J(X;\eta) = \int_{[1,X]^3} \widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 - \mu)\, d\mathbf{u}. \tag{18}$$

Let U denote the set of (u_1, u_2) for which

$$X/2 \le \lambda_1 u_1 \le 2X/3 \quad \text{and} \quad X/4 \le |\lambda_2| u_2 \le X/3.$$

If X is taken large enough so that $|\mu| \le X/12$, then for any $(u_1,u_2) \in U$, we have $X/12 \le \lambda_1 u_1 + \lambda_2 u_2 - \mu \le X/2$. Hence for each point in U there is an interval $V(u_1,u_2)$ for u_3, of length η/λ_3 and contained in $[1,X]$, on which the integrand in (18) is at least $\eta/2$. It follows that $J(X;\eta) \gg \eta^2 X^2$ and therefore

$$I_{\mathfrak{M}}(X;\eta) \gg \eta^2 X^2. \tag{19}$$

We divide the remaining region into minor and trivial arcs as follows. In view of Lemma 2, we can find a non-decreasing sequence of positive numbers X_m, tending to infinity, such that

$$\sup_{LX^{-1}\le|\alpha|\le m} |f_1(\alpha)f_2(\alpha)| \le m^{-1}X^2$$

whenever $X > X_m$. Now define the function $T(X)$ by taking $T(X) = m$ when $X_m < X \le X_{m+1}$, and write

$$\mathfrak{m} = \{\alpha : LX^{-1} < |\alpha| \le T(X)\}$$

and

$$\mathfrak{t} = \{\alpha : |\alpha| > T(X)\}.$$

We seek to demonstrate that the contribution from each of these regions is $o(X^2)$. First of all, we have by (15) and the trivial inequality $xyz \le x^3 + y^3 + z^3$ that

$$I_{\mathfrak{t}}(X;\eta) \ll \sum_{n>T(X)} n^{-2} \int_{n-1}^{n} |f_i(\alpha)|^3 \, d\alpha$$

for some index i. Thus, after a change of variable, it follows from Lemma 3 and periodicity that

$$I_{\mathfrak{t}}(X;\eta) \ll \frac{X^2}{T(X)}, \tag{20}$$

and this bound is indeed $o(X^2)$ since $T(X) \to \infty$. For the minor arcs, the trivial inequality $x^3y^3z^4 \le x^{10} + y^{10} + z^{10}$ yields

$$I_{\mathfrak{m}}(X;\eta) \ll \sup_{\alpha\in\mathfrak{m}} |f_1(\alpha)f_2(\alpha)|^{1/4} \int_{\mathfrak{m}} |f_i(\alpha)|^{5/2} K(\alpha)\, d\alpha$$

for some index i. But by construction we have

$$\sup_{\alpha\in\mathfrak{m}} |f_1(\alpha)f_2(\alpha)| = o(X^2),$$

so on making a change of variable and applying Lemma 3 and (15) as in the treatment of $\mathfrak{t}$, we find that

$$I_{\mathfrak{m}}(X;\eta) = o(X^2). \tag{21}$$

Theorem 1 now follows immediately on collecting (19), (20), and (21).

4. Positive definite binary forms

The analytic set-up used to establish Theorem 2 is slightly different from that of the previous section, but the analysis itself requires essentially the same techniques. Let $\eta \in (0,1)$, let $\mathcal{Z}$ be an η-spaced set of real numbers, and write $R(z;\eta)$ for the number of solutions of the inequality (2) in primes p_1 and p_2. We may again suppose after suitable rescaling that $\lambda_1 \ge 1$ and $\lambda_2 \ge 1$, since the set $c\mathcal{Z}$ is obviously $c\eta$-spaced. Let X be a positive real number, sufficiently large in terms of $\boldsymbol{\lambda}$, and define the set

$$\mathcal{E}^* = \mathcal{E}^*(X;\mathcal{Z};\eta) = \{z \in (X/2, X] \cap \mathcal{Z} : R(z;\eta) = 0\}.$$

We introduce the exponential sum

$$H(\alpha) = \sum_{z \in \mathcal{E}^*} e(-\alpha z),$$

and let $K(\alpha)$ be as in (13). As in §3, it follows from (14) that the integral

$$\int_{-\infty}^{\infty} f_1(\alpha) f_2(\alpha) e(-\alpha z) K(\alpha)\, d\alpha$$

is zero unless the inequality (2) has a solution. We therefore have

$$\int_{-\infty}^{\infty} f_1(\alpha) f_2(\alpha) H(\alpha) K(\alpha)\, d\alpha = 0, \tag{22}$$

and this relation provides the starting point for our analysis.

We use the exact same definitions of $\mathfrak{M}$, $\mathfrak{m}$, and $\mathfrak{t}$ as in §3 to analyze the various contributions to this integral. We shall also find it convenient to write $\mathfrak{m}^* = \mathfrak{m} \cup \mathfrak{t}$. By following the analysis of the previous section, one sees immediately that

$$\int_{\mathfrak{M}} f_1(\alpha) f_2(\alpha) H(\alpha) K(\alpha)\, d\alpha = \sum_{z \in \mathcal{E}^*} \left(J(z; X; \eta) + O(X (\log X)^{-B}) \right),$$

where

$$J(z; X; \eta) = \int_1^X \int_1^X \widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 - z)\, du_1\, du_2.$$

Whenever $z \in (X/2, X]$ and $\lambda_1 u \in [z/4, z/2]$, there is a corresponding interval of length η/λ_2 for u_2, contained in $[1, X]$, on which the integrand above is at least $\eta/2$. Hence $J(z; X; \eta) \gg \eta^2 z$, and

$$\int_{\mathfrak{M}} f_1(\alpha) f_2(\alpha) H(\alpha) K(\alpha)\, d\alpha \gg \eta^2 XY,$$

where we have written $Y = \operatorname{card}(\mathcal{E}^*)$. By (22) and the Cauchy-Schwarz inequality, we may conclude that

$$\eta^2 XY \ll \left| \int_{\mathfrak{m}^*} f_1(\alpha) f_2(\alpha) H(\alpha) K(\alpha)\, d\alpha \right| \ll J_1^{1/2} J_2^{1/2}, \tag{23}$$

where

$$J_1 = \int_{\mathfrak{m}^*} |f_1(\alpha) f_2(\alpha)|^2 K(\alpha)\, d\alpha$$

and

$$J_2 = \int_{-\infty}^{\infty} |H(\alpha)|^2 K(\alpha)\, d\alpha.$$

Now fix $\varepsilon \in (0,1)$. By (15), we have $K(\alpha) \ll \eta^{1-\varepsilon} \alpha^{-1-\varepsilon}$, so on considering separately the sets $\mathfrak{m}$ and $\mathfrak{t}$ and making use of Lemmas 2 and 3 as in the argument at the end of §3, we find that

$$\eta^{-1+\varepsilon} J_1 = o(X^3), \tag{24}$$

where the implicit function depends at most on ε and $\boldsymbol{\lambda}$. Moreover, in view of the spacing condition on $\mathcal{Z}$, we have by (14) that

$$J_2 = \sum_{z_1, z_2 \in \mathcal{E}^*} \widehat{K}(z_1 - z_2) = \eta Y.$$

Substituting this into (23) gives

$$\eta^2 XY \ll (J_1 \eta Y)^{1/2},$$

and thus (24) yields

$$\eta^{2+\varepsilon} Y \ll \eta^{-1+\varepsilon} J_1 X^{-2} = o(X).$$

The bound for $E(X;\eta)$ claimed in Theorem 2 now follows by summing over dyadic intervals.

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